On Sums of Powers

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$$S_3(n) = 1^3 + 2^3 + \cdots + n^3 = \tfrac{n^2(n+1)^2}{4}, \qquad \textit{Aryabhata}(476AD)$$

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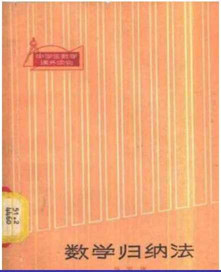

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$$S_k(n) = \frac{1}{k+1}\sum_{j=0}^{k}(-1)^j C_j^{k+1} B_j n^{k+1-j} = \frac{(n-B)^{k+1}-(-B)^{k+1}}{k+1}.$$

Yang Hui – Pascal triangle

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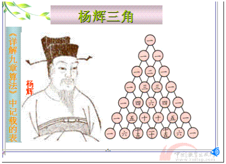

Yang Hui (1238–1298)

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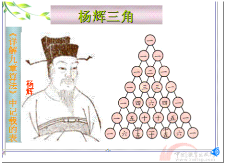

$$n - (n - 1) = 1$$

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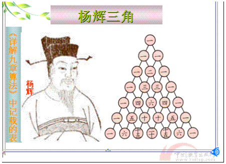

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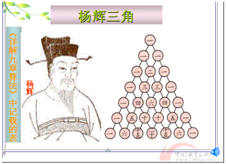

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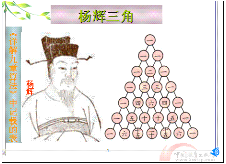

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$$c_{k1}+c_{k2}n+c_{k3}n^2+\cdots$$

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$$1^k + 2^k + \cdots + n^k = a_{k1} n + a_{k2} n^2 + \cdots + a_{kk} n^{k+1}$$

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proof

Set $G(z, n) = \sum_{k=0}^{\infty} S_k(n) \frac{z^k}{k!}$.

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Set $G(z,n)=\sum_{k=0}^{\infty} S_k(n)\frac{z^k}{k!}$. Then change order of sums to obtain

$$G(z,n)=\frac{1-e^{nz}}{e^{-z}-1}=\sum_{j=0}^{\infty} B_j \frac{(-z)^{j-1}}{j!}\sum_{i=1}^{\infty}-\frac{(nz)^i}{i!}$$

$$=\sum_{k=0}^{\infty}\frac{z^k}{k!}\frac{1}{k+1}\sum_{j=0}^{k}(-1)^j C_{k+1}^j B_j n^{k+1-j}.$$

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For any nice function $f(x)$, the sum $S(n) = \sum_{i=1}^{n} f(i)$ extends to a nice function $S(x)$.

Euler formula

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then

$$f(x) = (1-e^{-D})S(x) = \frac{1-e^{-D}}{D}DS(x)$$

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For $f(x) = x^k$ with $k \geq 0$ an integer, Euler's formula recovers Bernoulli's formula.

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$$S_{-k}(n) = S_{-k}(\infty) + \frac{1}{(k-1)n^{k-1}} + \frac{1}{2n^k} + \cdots, \qquad (k > 1)$$

Question

For each real $k > 1$, how to evaluate the infinite sum:

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In 1735, Euler made the first major contribution to this problem: he solved the case $k = 2$ which was called Basel problem:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \approx 1.6449.$$

He observed the fact $\sin x = 0$ if and only if $x = 0, \pm\pi, \pm 2\pi \cdots$ which leads to a product formula:

$$\sin x = x \prod_{k=1}^{\infty} (1 - \frac{x^2}{k^2 \pi^2}).$$

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On the other hand, there is also an addition formula:

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Comparing the coefficients of x^3 in these two expressions gives

$$\sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} = \frac{1}{3!}.$$

The argument also applies to symmetric functions of $1/\pi^2 k^2$ and thus gives a formula for $\zeta(2k)$. More precisely, two expansions of $\cot x = (\log \sin x)'$ will give

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Euler extended $\zeta(s)$ to reals in $(0,1)$ by the following methods:

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It has an asymptotic behavior near $s=1$:

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Euler define infinite sum of non-negative powers by

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In 1739, Euler evaluated the values of ζ at negative integers and obtained

$$\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1}, \quad k < 0$$

$1 + 1 + \cdots + 1 \cdots = -\frac{1}{2}$
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$1 + 2^4 + 3^4 \cdots + \cdots + 100^4 + \cdots = 0$

Euler proved his formula by the following expression:

$$
\begin{aligned}
\zeta(-k) =& 1 + 2^k + 3^k + \cdots \\
=& \frac{1}{1-2^{k+1}}(1 - 2^k + 3^k + \cdots) \\
=& \frac{1}{1-2^{k+1}} \lim_{x \to 1^-} (x - 2^k x^2 + 3^k x^3 + \cdots) \\
=& \frac{1}{1-2^{k+1}} \left(x \frac{d}{dx} \right)^k \Big|_{x=1} \frac{x}{1+x}.
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In comparison with $\zeta(2k) = \frac{(-1)^{k-1} B_{2k} (2\pi)^{2k}}{2(2k)!}$, there is a "functional equation" between $\zeta(s)$ and $\zeta(1-s)$ for integers.

Arithmetic progress of primes

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Dirichlet (1805–1859)

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For two integers N, a coprime to each other, there are infinitely many primes p such that $p \equiv a \mod N$.
Moreover,

$$\sum_{p \equiv a \mod N} \frac{1}{p} = \infty.$$

Fourier transform of periodic functions

Fourier (1768–1830)

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Every periodic and continuous function $f(x) = f(x+1)$ has a spectral decomposition:

$$f(x) = \sum_{n=-\infty}^{\infty} \widehat{f}(-a)e^{2\pi iax}$$

$$\widehat{f}(a) := \int_0^1 f(x)e^{2\pi iax}dx.$$

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Every square integrable function on $\mathbb{R}$ has a decomposition:

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Every integrable function on $(0, \infty)$ for measure dt/t has a decomposition:

$$f(x) = \int_{-i\infty}^{i\infty} \widehat{f}(-s) x^s \frac{ds}{2\pi i}$$

$$\widehat{f}(s) = \int_0^{\infty} f(t) t^s \frac{dt}{t}.$$

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Every function on $\{0, \cdots, N-1\}$
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Multiplicative version on $(\mathbb{Z}/N\mathbb{Z})^\times$:
Every function on prime to N numbers in $\{0, \cdots, N-1\}$

$$f(x) = \sum_{\chi:\text{characters}} f(\chi^{-1}) \chi(x)$$

$$\widehat{f}(\chi) = \frac{1}{\phi(N)} \sum_{(\mathbb{Z}/N\mathbb{Z})^\times} f(x) \chi(x).$$

Fourier analysis on multiplicative group

Apply to $G = (\mathbb{Z}/N\mathbb{Z})^\times$ and $f = \delta_a$ (Dirac delta function), we obtain

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When $\chi = 1$, $\sum_p \frac{\chi(p)}{p} = \infty$ by Euler, it suffices to show when $\chi \neq 1$

$$\sum \frac{\chi(p)}{p} \neq \infty$$

Dirichlet L-function

For χ a character of $(\mathbb{Z}/N\mathbb{Z})^\times$, Dirichlet introduce his function

$$L(\chi, s) = \sum_n \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1 - \frac{\chi(p)}{p^s}}, \qquad s > 1.$$

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Then

$$\log L(\chi, 1) = \sum \frac{\chi(p)}{p} + O(1).$$

So is suffices to show that $L(\chi, 1)$ is finite and nonzero.

Mixing addition and multiplication

Now we apply additive Fourier expansion χ: $\chi = \sum_{a=0}^{N-1} \widehat{\chi}(e_{-a})\psi_a$ to obtain

$$L(\chi, s) = \sum_{a=0}^{N-1} \widehat{\chi}(\psi_{-a})L(\psi_a, s), \quad L(\psi_a, s) = \sum_n \frac{e^{2\pi i a n/N}}{n^s}$$

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$$L(e_a, 1) = \sum_{n=1}^{\infty} \frac{e_a(1)^n}{n} = -\log(1 - e_a(1)).$$

Dirichlet formula

Combining everything gives an important special value formula:

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Example: $N = 4$, $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \to \{\pm 1\}$, then

$$L(\chi, 1) = 1 - \frac{1}{3} + \frac{1}{5} \cdots = \frac{\pi}{4} \approx 0.7853.$$

Riemann's memoire

In 1859, in his memoire "on the number of primes less than a given quantity", consider zeta function with complex variable $\mathrm{Re}(s) > 1$:

Riemann (1826–1866)

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Riemann discovered several extremely important properties using **Fourier analysis** on $\mathbb{R}$ and $\mathbb{R}^\times$.

Expression in Fourier transform

Using $\Gamma(s) := \langle e^{-x}, x^{s-1}\rangle = \int_0^\infty e^{-x}x^s\frac{dx}{x}$, one obtains $\zeta(s)$ as the Mellin transform of a theta function:

$$\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s) = \int_0^\infty f(t)t^{s/2}\frac{dt}{t}, \qquad f(t) := \sum_{n=1}^\infty e^{-\pi n^2 t}.$$

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In terms of multiplicative Fourier transfer (or Mellin transfer)

$$\xi(s) = \widehat{f}(s/2).$$

Continuation and functional equation

Writing $\int_0^\infty = \int_0^1 + \int_1^\infty$ and using Poisson summation formula, we have

$$\xi(s) = \int_1^\infty f(t)(t^{s/2} + t^{(1-s)/2})\frac{dt}{t} - \left(\frac{1}{s} + \frac{1}{1-s}\right).$$

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This gives the meromorphic continuation and the functional equation $\xi(s) = \xi(1-s)$. This somehow explains the relation between $\zeta(1-k)$ and $\zeta(k)$ given by Euler.

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There is another product formula of $\zeta(s)$ in terms of its zeros (as $\sin x$).

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Then Riemann bravely made a hypothesis that all zeros of $\xi(x)$ lies on the line $\mathrm{Re}(s) = 1/2$.
Riemann Hypothesis is equivalent to an asymptotic formula for the number of primes

$$\pi(x) := \#\{p \le x\} = \int_2^x \frac{dt}{\log t} + O_\epsilon(x^{\frac{1}{2}+\epsilon})$$

which is a conjectural stronger form of the prime number theorem.

L-functions

A general L-functions takes a form

$$L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \prod_p \frac{1}{F_p(s)} \qquad a_n \in \mathbb{C}.$$

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How to find a general L-series?

Review Euler and Riemann

$$\prod_p \frac{1}{1-\frac{1}{p^s}} \overset{Euler}{=} \zeta(s) \overset{Riemann}{=} \frac{\pi^{s/2}}{\Gamma(s/2)} \int_0^\infty f(t) t^{s/2} \frac{dt}{t}.$$

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The left hand side is pure arithmetic as it reflects distribution of primes $\{2, 3, 5, \cdots\}$, and right hand side is purely analytic as it decomposition into multiplicative spectrum of $f(x) = \sum_{n=1}^\infty e^{-\pi n x^2}$.

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These two constructions can be generalized to high dimensions situation with counting points on algebraic varieties and spectral decompositions on functions on Lie groups.

We usually call them motivic L-functions and automorphic L-functions respectively.

Langlands program

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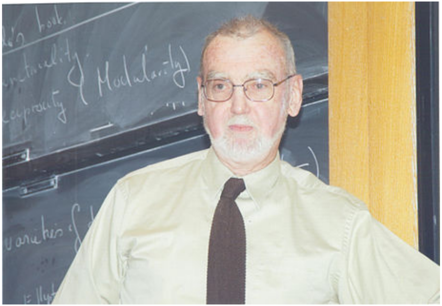

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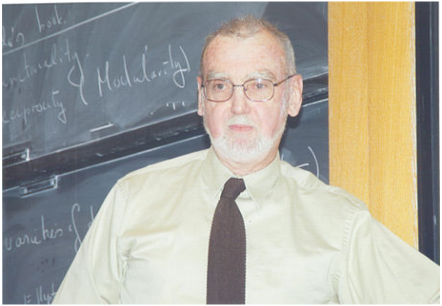

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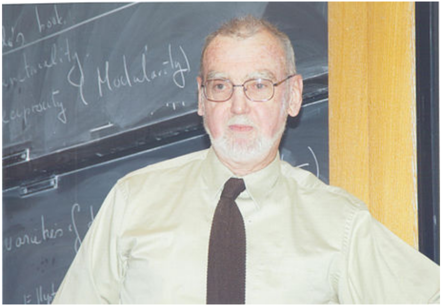

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The program connects two different worlds of mathematics: "arithmetic" vs "harmonic analysis".

Ramanujan τ-function

In 1917, Ramanujan studied the function $q\prod_{n=1}^{\infty}(1-q^n)^{24}$. With $q = e^{2\pi iz}$, the above defines a function $\Delta(z)$ with $\mathrm{Im}z > 0$.

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Ramanujan (1887–1920)

He conjectured and later proved by Mordell that this function is an eigen modular form of weight 12:

$$\Delta\left(\frac{az+b}{cz+d}\right)=(cz+d)^{12}\Delta(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z})$

Hecke L-functions

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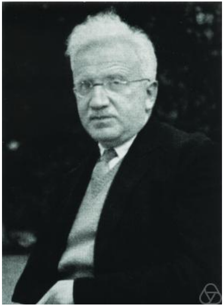

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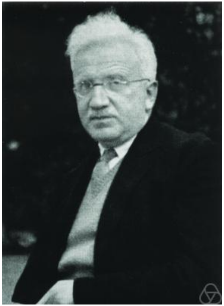

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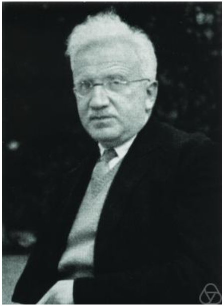

Enrich Hecke (1887–1947)

He showed that each $L(f,s)$ has a holomorphic continuation with functional equation.

Furthermore, he introduced operators T_n on S_k and showed that if f is an eigen form, then we have an Euler product

$$L(f,s) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}$$

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As a consequence, he proved the Peterson–Ramanujnan's conjecture

$$|a_p| \leq p^{(k-1)/2}.$$

L-functions for Elliptic curves

The simplest but non-trivial motivic L-functions are those coming from elliptic curves

$$E : y^2 = x^3 + ax + b, \quad a, b \in \mathbb{Z}, 4a^3 + 27b^2 \neq 0.$$

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where for finitely many p, $\epsilon(p) = 0$, $a_p = \pm 1, 0$.
Otherwise, $\epsilon_p = 1$ and $1 - a_p + p$ is the number of solutions of

$$y^2 \equiv x^3 + ax + b \mod p.$$

Modularity theorem

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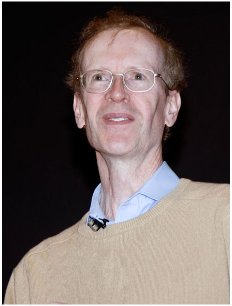

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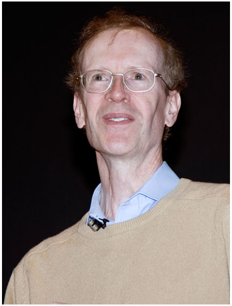
Modularity theorem

In 1994, Wiles proved the modularity theorem $L(E.s) = L(f,s)$ for semistable elliptic curves. The full modularity theorem was proved in 2001.



Andrew Wiles (1953–)

As a consequence, Wiles proved the Fermat last theorem: there are no positive integers $a, b, c, n \geq 3$

$$a^n + b^n = c^n.$$

Special values of L-series

A web of conjectures assert that the special values of motivic L-functions often give crucial information about the Diophantine properties of the varieties, such as BSD, Tate, etc.

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For elliptic curves, the most notable work were done by Gross–Zagier in 1980's and by
Zhiwei Yun and Wei Zhang in 2015.

Future

Riemann hypothesis, Langlands program, and special values of L-series are three major topics of number theory in the 21st century.