Regulator for the Characteristic Variety and Volume Conjecture

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National Tianyuan Fundation
2018 Southwest Mathematics Center
September 12, 2018

Low dimensional Topology

Low-dimensional topology is to study manifolds, or more generally topological spaces, of four or fewer dimensions.

Smale in 1961 proved the Poincaré conjecture in higher dimensions ≥ 5 and made dimensions three and four seem the hardest;

Thurston's geometrization conjecture, formulated **in the late 1970s**, offered a framework that suggested geometry and topology were closely intertwined in low dimensions; **In 2002 Grigori Perelman** announced a proof of the three-dimensional Poincar conjecture;

In early 1980s, Vaughan Jones' discovery of the Jones polynomial not only led knot theory in new directions but gave rise to still mysterious connections between low-dimensional topology and mathematical physics.

Simon Donaldson in 1982 "stunned the mathematical world" (Atiyah 1986) (Exotic Structure) **Michael Freedman in 1982** proved the 4-dimensional Generalized Poincar conjecture. Freedman and Kirby showed that an exotic $\mathbf{R}^4$ manifold exists. An exotic $\mathbf{R}^4$ is a differentiable manifold that is homeomorphic but not diffeomorphic to the Euclidean space $\mathbf{R}^4$.

Differential Geometry, Topology, Analysis, Algebraic Geometry, Algebra, Nonlinear PDE, Dynamic System, Representation Theory, etc.

ENCOURAGE more younger people to study low dimensional topology!

OUTLINE

(i) 3–Manifold and Knot topology:

Volume conjecture,

Witten's TQFT approach,

Gukov's complexification;

(ii) Algebraic Geometry method:

$SL_2(C)$ character variety

Beilinson Regulator of curves

K_2 group for curves

Bohr-Sommerfeld Quantization

Reformulate the volume conjecture and discussions

Part I. The volume conjecture and TQFT

§0. Background for the volume conjecture

Kashaev (1997) defined a series of invariants of a link by using the quantum dilogarithm.

H. Murakami and J. Murakami (2001) identified Kashaev invariants with the $N-$colored Jones polynomial $J_N(K, e^{2\pi i/N})$ of the link evaluated at $e^{2\pi i/N}$.

Volume Conjecture: *For any knot K,*

$$2\pi \lim_{N\to\infty} \frac{\log |J_N(K, e^{2\pi i/N})|}{N} = v_3 \|S^3 \setminus K\|$$

where $\|\cdot\|$ is the simplicial volume and v_3 is the hyperbolic volume of the regular ideal tetrahedron.

True for the torus knots, figure eight, some Whitehead doubles of torus knots and other knots by *straightforward* calculations.

Knot invariants and TQFT

Jones polynomial can be defined purely from representations of Hecke algebra, or skein module combinatorically, or Categorification..

Finding a geometric and topological definition or relation is one of most important questions in knot theory.

A knot is trivial iff both its L^2-torsion and its Alexander polynomial are trivial.

A knot is trivial iff every Vassiliev finite type invariant of the knot agrees with the one of the trivial knot.

The colored Jones polynomials and Alexander polynomial are determined by the Vassiliev finite type invariant.

In short, Jones polynomial detects the trivial knot if the volume conjecture is true.

§1. Witten's TQFT and Jones invariant

Stationary Phase Approximation

Let $f : M \to R$ be a C^2 Morse function on the n-dimensional manifold M.

$$Z(M) = \int_M e^{ikf(y)} dy =$$

$$\left(\frac{2\pi}{k}\right)^{n/2} \sum_{df(x)=0} \frac{e^{ikf(x)} e^{\pi i sgn(H(f)(x))/4}}{\sqrt{H(f)(x)}}$$

$$+O(k^{-n/2-1})$$

as $k \to \infty$, where $H(f)(x)$ is the Hessian of f at the critical point x.

Witten's TQFT

Let $f = cs$ and $M = \mathcal{B}_Y$ the space of gauge equivalence classes of connections of a principle $SU(2)$ bundle over a closed 3–manifold Y^3.

$$Z(Y) = \int_{\mathcal{B}_Y} e^{ikcs([a])} d[a]$$

$$\sim \sum_{F_a=0} e^{ikcs(a)} \frac{e^{\pi i sgn(*d_a)/4} \det(d_a^* d_a)^{1/2}}{|\det(*d_a)|^{1/2}}$$

$$= \sum_{F_a=0} e^{ikcs(a)} e^{\pi i \eta(a)/4} \sqrt{T(a)}$$

as $k \to \infty$, where $\eta(a)$ is metric dependent eta invariant and $T(a)$ is the Reidemeister–Ray–Singer torsion of the flat connection a.

Witten's invariant from TQFT is a weighted sum of topological invariant.

The colored Jones invariant

$$J_N(K, e^{2\pi i(k_0+2)}) = \langle W_{R_j(K)} \rangle$$

is the expectation of Wilson loop observables.

The trivial connection is flat and has a contribution: By Milnor and Turaev: $\sqrt{T(a)} = \frac{2\sin(\pi t)}{\nabla(K, e^{2\pi t})}$ for $t = N/k$ the $U(1)$ holonomy around the Wilson loops.

$$Z_{SU(2)}(S^3) = \sqrt{\frac{2}{k}} \sin(\frac{\pi}{k}).$$

$$Z^{tr}_{SU(2)}(W_{R_j(K)}) \sim \sqrt{\frac{2}{k}} \frac{\sin(\pi t)}{\nabla(K, e^{2\pi t})},$$

$$J^{tr}_N(K, e^{2\pi i/k}) \sim \frac{k \sin(\pi t)}{\pi \nabla(K, e^{2\pi t})},$$

The normalized Jones invariant gives $\sim \frac{1}{\nabla(K, e^{2\pi t})}$ which implies the Melvin–Morton conjecture (Rozansky's approach)

§2. Gukov's complexification

Let f be the Chern–Simons functional over $\mathcal{B}_Y$ with $G = SL_2(C)$, $Y = S^3 \setminus K$.

$$Z(Y) \sim \sum_{F_a=0} e^{ikcs(a)} e^{\pi i \eta(a)/4} \sqrt{T(a)}$$

as $k \to \infty$, where the "sum" is over the $SL_2(C)$ flat connection.

The contribution of the hyperbolic flat connection a_h:

$$\log J_N^{a_h}(K, q) \sim \frac{N}{2\pi}(Vol(a_h) + i2\pi^2 cs(a_h))$$

gives the **generalized Volume Conjecture**.

$SL_2(C)$ flat connections: $X(S^3 \setminus K)$ is of 1–complex dimension.

Question: Is there natural topological invariant parametrized by C^* and related to the Alexander–type polynomial as volume ?

$X(S^3 \setminus K)$ is related to the A-polynomial studied by [CCGLS] (Cooper, Culler, Gillet, Long, Shalen, Invent. Math 1994).

Gukov interpreted the A-polynomial zero locus as a Lagrangiang subspace in the diagnoalization part $C^* \times C^*$ of $X(T^2)$.

So there is a complex 1-form θ defined on the Lagrangian subspace.

Quantization conditions: proposed by Gukov,

(1) $\int_C Im(\theta) = 0$;

(2) $\frac{1}{(2\pi)^2} \int_C Re(\theta)$ is rational,

for every closed loop in zero-locus of the A-polynomial of the hyperbolic knot.

It is important to have the Bohr-Sommerfeld quantization condition to have the system consistently quantized and to have semi-classical expression for the partitions formula.

Part II. Algebraic Geometry Method, Regulator and K_2

§3 $SL_2(C)$ character variety

Let K be a hyperbolic knot in S^3, and M_K be the hyperbolic 3-manifold with finite volume.

$R(M_K) = Hom(\pi_1(M_K), SL_2(C))$, and $t : R(M_K) \to X(M_K)$ be the canonical surjective morphism.

$R(\partial M_K) = \{(A, B) | A, B \in SL_2(C), AB = BA\}$ with $A = \rho(\mu), B = \rho(\lambda)$ and (λ, μ) fixed generators in π_1.

$R_D \subset R(\partial M_k)$ consists of ρ's with $\rho(\mu) = diag(m, m^{-1}), \rho(\lambda) = diag(l, l^{-1})$.

$R_D \cong C^* \times C^*$

$\chi \in X(\partial M_K)$ is determined by its values on $\mu, \lambda, \mu\lambda$.

Define $t : R(\partial M_K) \to C^3$ by

$$t(\rho) = (tr(\rho(\mu)), tr(\rho(\lambda)), tr(\rho(\mu\lambda))).$$

Then $X(\partial M_K) = t(R(\partial M_K))$.

$t_D = t|_{R_D}$ is given by $(m + m^{-1}, l + l^{-1}, ml + m^{-1}l^{-1})$

Let ρ_0 be the discrete faithful representation corresponding to the hyperbolic structure.

R_0: the irreducible component of $R(M_K)$ containing ρ_0

$X_0 = t(R_0)$. $X_0 \subset X(M_k)$ is an irreducible affine variety of dimension 1.

$t : R_0(\subset R(M_K)) \to X_0(\subset X(M_k))$

$r : X_0 \to Y_0 = \overline{r(X_0)} \subset X(\partial M_K)$

$t_D : D_0 = t_D^{-1}(Y_0)(\subset R_D) \to Y_0.$

D_0 is an affine algebraic set of dimension 1;

The image of each 1-dimensional component of D_0 under t_D is the whole Y_0.

D_0 has no 0-dimensional components, has at most two 1-dimensional components.

The A-polynomial $A(l, m)$ is the defining polynomial of the closure of the union of D_0's (from other irreducible component $Y^{'}$ like Y_0).

§4 Beilinson Regulator of curves

Background on Regulator

Regulator lies in the area of number theory and arithmetic geometry.

(i) Dirichlet Theorem: Let F be a number field with $n = [F : Q] = r_1 + 2r_2$. $r : O_F^* \to R^{r_1+r_2}$ (O_F^* group of units of the integer ring).

Imr as a lattice in he hyperplane $\sum_{i=1}^{r_1+r_2} y_i = 0$.

Covolume $R_D = vol(H/r(O_F^*))$ is the Dirichlet regulator.

$$\lim_{s\to 0} s^{-(r_1+r_2-1)}\zeta_F(s) = -\frac{h_F R_D}{\omega_F}$$

for class number h_F and ω_F is the number of roots of unity in F.

(ii) Borel Theorem: $r(m) : K_{2m-1}(F) \to R^{d_m}$

Borel regulator $R_m(F) = covolume(Imr(m) in H)$.

When $m = 1$, $K_1(F) = O_F^*$ and $R_1(F) = R_D$,

Think F as the 0-dimensional variety $Spec(F)$, Next is the algebraic curve (1-dimensional variety) case, it was done by Bloch, Beilinson and Deligne independently.

Beilinson regulator construction

Let X be a smooth projective curve over C.

Let f, g be meromorphic functions on X.

Let $S(f)$ be the set of zeros and poles of f.

Beilinson defined an element $r(f,g) \in H^1(X^{'}; C^*)$, where $X^{'} = X \setminus (S(f) \cup S(g))$ by

$r(f,g)(\gamma) = \exp(\frac{1}{2\pi i}\left(\int_\gamma log f \frac{dg}{g} - \log g(t_0) \int_\gamma \frac{df}{f}\right)),$

where γ is a loop in $X^{'}$ and t_0 is a distingushed base point in $X^{'}$.

Facts: (a) $r(f,g)(\gamma)$ is independent of the based point t_0.

(b) $r(f,g)$ is independent of the branches of $\log f$ and $\log g$.

(c) Deligne showed that $H^1(X^{'};C^*)$ is the group of isomorphism classes of the line bundle over $X^{'}$ with flat connection.

(d) the curvature of the line bundle associated to $r(f,g)$ is $\frac{df}{f} \wedge \frac{dg}{g}$.

(e) $r(f_1 f_2, g) = r(f_1, g) \otimes r(f_2, g)$

(f) $r(f,g) = r(g,f)^{-1}$

(h) Steinberg relation $r(f, 1-f) = 1$ for $f \neq 0, 1$.

(i) If $x \in S(f) \cup S(g)$ and γ_x is a small loop around x in $X^{'}$, then $r(f,g)(\gamma_x)$ is the tame symbol $T_x(f,g)$ of f,g at x.

§5 K_2 group for curves

Let $C(X)$ be the field of meromorphic functions on X, and $C(X)^*$ be the set of non-zero meromorphic functions on X.

Matsumoto Theorem:

$$K_2(C(X)) = \frac{C(X)^* \otimes C(X)^*}{\langle f \otimes (1-f) : f \neq 0, 1 \rangle},$$

where the tensor product is taken over integer Z, the denominator means the subgroup generated by those elements.

By Facts (e), (f), (h), we have $r(\{f, g\}) = r(f, g)$:

$r : K_2(C(X)) \to H^1(X \setminus S; C^*)$

Let Y be an irreducible component of the zero locus of the A-polynomial $A(l, m)$.

Proposition (Li-Wang): The element

$$\{l, m\} \in K_2(C(Y))$$

is a torsion element.

Suppose the component Y $(= X_0)$ contains y_0 which corresponds to the discrete faithful character of the hyperbolic structure and $m(y_0) = 1$.

Base point: if y_0 is a smooth point, choose $t_0 = y_0$; otherwise we fix a point in the preimage of y_0 in $\tilde{Y}$ (the smooth projective model of Y) and choose t_0 as this fixed point (equivalent to fixing a branch at the singular point y_0).

Proposition (Li–Q. Wang)

Over the character variety X_0 (normal curve), from the Beilinson regulator map,

$$2\pi i \log r(l,m) = \int_\gamma \log l \frac{dm}{m} - \log m(t_0) \int_\gamma \frac{dl}{l}$$

has imaginary part

$$\int_\gamma \eta(l,m) = \int_\gamma \log |l| d\arg m - \log |m| d\arg l$$

and the real part

$$\int_\gamma \xi(l,m) = -\int_\gamma (\log |m| \cdot d \log |l| + \arg l \cdot d\arg m).$$

(i) $r(l,m) \in H^1(X_0, \mathbf{C}^*)$ is a torsion.

(ii) the closed 1-form $\eta(l,m)$ is exact on X_0;

(iii) $\frac{1}{(2\pi)^2} \int_\gamma \xi(l,m) \in \frac{1}{N}\mathbf{Z}$, where N is the order of the symbol $\{l,m\}$ in $K_2(\mathbf{C}(Y))$.

Outline of the proof

(i) There is a finite field extension F of $C(Y)$ such that $\{l, m\} \in K_2(F)$ is of order at most 2.

Inclusion map $i : K_2(C(Y)) \to K_2(F)$ and the transfer map $t_{K_2} : K_2(F) \to K_2(C(Y))$

The composition $t_{K_2} \circ i$ is given by multiplication of $n = [F : C(Y)]$ the degree of the finite extension.

$2t_{K_2} \circ i(\{l, m\}) = t_{K_2}(2i(\{l, m\})) = 0$ and $= 2n\{l, m\}$.

(ii) For any loop γ in the smooth part,

$$r(l,m)(\gamma)^q = 1$$

due to torsion property.

Write $r(l,m) = \exp\left(\frac{1}{2\pi i}(Re + iIm)\right)$. Then we have

$Im = 0$ and $\frac{q \cdot Re}{2\pi i} = 2\pi i p$ for some integer p.

(ii) and (iii) follow from the identifications of Re and Im parts.

Remarks: (1) Gukov argued (ii) and (iii) as quantized condition from math-physics. It gives a stronger version of Bohr-Sommerfeld quantization with more information on the rational number.

(2) $\eta(l,m) = \frac{1}{2} dVol$ follows from Hodgson, $\xi(l,m) = d\text{"}cs\text{"}$. The form ξ is not the same Chern–Simons from Kirk–Klassen derived from c_2 second Chern class.

(3) Bloch Conjecture that $c_i \in H_D^{2i}(X, Z(i))$ in Deligne cohomology of C^∞ complex projective variety X is torsion for $i \geq 2$ and holomorphic flat vector bundle. (A. Reznikov confirmed the conjecture)

(4) Question: Does the result hold for other irreducible components Y' which does not containing $t_0 = t(\rho_0)$?

Theorem (Li–Q. Wang)

Under the regulator map $K_2(\mathbf{C}(Y)) \to H^1(Y, \mathbf{C}^*)$,

(1) $\{l, m\}$ is mapped into $\exp(\frac{1}{2\pi i}(\xi(l, m) + i\eta(l, m))) = r(l, m)$ in $H^1(Y, \mathbf{C}^*)$. Note that ξ is only well-defined up to $\frac{1}{2\pi} d arg m$.

(2) The line bundle constructed from Bloch etal is pull–back from the universal Heisenberg line bundle with connection on $\mathbf{C}^* \times \mathbf{C}^*$, over $Y = X \backslash$ zeros and poles of l, m.

(3) The curvature of the line bundle is $\frac{dl}{l} \wedge \frac{dm}{m}$ and

$$d(\xi(l, m) + i\eta(l, m)) = \frac{dl}{l} \wedge \frac{dm}{m} = 0.$$

Remarks: (i) Note that $H^1(X', \mathbf{C}^*)$ is the group of isomorphism classes of the line bundle over X' with a flat connection associate to the class in $H^1(X', \mathbf{C}^*)$ viewed as $\pi_1(X') \to \mathbf{C}^*$.

(ii) Thus the $\frac{1}{2\pi i}(\xi + i\eta)$ can be thought of as the Chern–Simons class from **the first Chern class** c_1 of the flat line bundle (not the usual transgression of c_2 class.

(iii) Any invariant arising from the zero locus of the A–polynomial may play a role in the volume conjecture. Qingxue Wang and I constructed some $SL_2(\mathbf{C})$–algebraic geometric invariant from the character variety X_0 for the hyperbolic knots.

See "On the generalized volume conjecture and regulator", to appear in Commun. Contemp. Math.

§6 Reformulate the volume conjecture from the aspect of regulator

For a path $c : [0,1] \to Y_h$ with $c(0) = t_0$ and $c(1) = (l,m)$, denote

$$U(l,m) = -q \int_c (log|y|dlog|x| + argxdargy).$$

Fix a number a with $m = -\exp(i\pi a)$, we reformulate the generalized volume conjecture:

$$\lim_{N,k\to\infty; N/k=a} \frac{log J_N(K, e^{2\pi i/k})}{k} =$$

$$\frac{1}{2\pi}(Vol(l,m) + i\frac{1}{2\pi}U(l,m)).$$

We have proved that $\frac{1}{(2\pi)^2}U(l,m)$ is well–defined in $\mathbf{R}/\mathbf{Z}$. The classical Chern–Simons invariant is well–defined in $\mathbf{R}/\mathbf{Z}$.

(i) Yoshida showed that there is an analytic function $F(u)$ that $|F(u)|$ is related to the volume and $argF(u)$ is related to the Chern–Simons of the hyperbolic 3–manifolds, by the Atiyah–Patodi–Singer index;

(ii) Dupont showed that there is an natural identification for the Chern–Simons class via the dilogarithm functions, from purely algebra point of view.

(iii) By focusing only on the regulator we have a different generalized volume conjecture from that of Gukov. Gukov and Murakami showed that their conjectures comes from the different choices of polarization.

related the regulator to other polarizations ?

motivic interpretation for the asymptotic Jones polynomials from Khovanov's work on categorification approach ?

Higher regulator for hyperbolic links

For a hyperbolic link $L \subset S^3$ with n components, we have an induced restriction map

$$r : X(M_L) \to X(T_1) \times \cdots \times X(T_n)$$

. Let $X_0 = t(R_0)$, where R_0 is the irreducible component of $R(M_L)$ containing the discrete faithful representation ρ_0 for the complete hyperbolic structure on M_L.

Proposition Let Y_0 be the Zariski closure of the image $r(X_0)$ in $X(T_1) \times X(T_2) \times \cdots X(T_n)$. Then Y_0 is an n-dimensional affine variety.

Define X_0^i be the subvareity of X_0 defined by $I^2_{\mu_j} - 4 = 0, j \neq i, 1 \leq j \leq n$ and V_i be an irreducible component of X_0^i containing $\chi_0 = t(\rho_0)$.

Proposition Let $r_i : X_0 \to X(T_i)$. Then we have V_i is of dimension one, and the Zariski Closure W_i of $r_i(V_i)$ in $X(T_i)$ has dimension one, for each i.

Let $R_D(T_i)$ be the subvariety of $R(T_i)$ which consists of diagonal representations. Let $t_i|_D$ be the restriction of t_i on $R_D(T_i) = C^* \times C^*$. Set $D_i = t_i^{-1}|_D(W_i)$.

Let $\overline{Y}_i$ be the smooth projective model of Y_i (an irreducible component of D_i containing y_i) and $\mathbf{C}(\overline{Y}_i)$ the function field of $\overline{Y}_i$.

There is an induced map on the K-groups:

$$j : \oplus_{i=1}^n K_2(\mathbf{C}(Y_i)) \to K_2(\mathbf{C}(Y^h)),$$

where $Y^h = \prod_{i=1}^n \overline{Y}_i$.

Proposition (i) The symbol $\sum_{i=1}^n (-1)^{\varepsilon(i)}\{m_i, l_i\}$ is a torsion element in $K_2(Y^h)$.

(ii) The higher holonomy of $\sum_{i=1}^n (-1)^{\varepsilon(i)}\{m_i, l_i\}$, is a torsion as representing higher order Deligne cohomology classes given by Gajer.

Hence the quantization condition for hyperbolic links holds from this higher regulator point of view.

§7 An approach to the volume conjecture from the L^2–invariant

Let $J_N(K,q)$ be the colored Jones polynomial.

Volume conjecture

$$\lim_{N\to+\infty}\left|J_N\left(K,\exp\left(\frac{2\pi\sqrt{-1}}{N}\right)\right)\right|^{\frac{1}{3N}}=\Delta_K^{(2)}(1).$$

(from the L^2 twisted Alexander invariant defined by Li–Zhang.)

Melvin–Morton conjecture*

$$\lim_{d\to+\infty}\frac{J\left(K,V_{d+1}\right)}{[d+1]}\left(\exp\left(\frac{h}{d}\right)\right)=\frac{1}{\Delta_K\left(\exp(h)\right)}.$$

(*) Has been proved by Rozansky, Bar–Natan and Garoufalidis (and also by Xiao–Song Lin and Zhenghan Wang).

Remark. (1) Viewing the volume conjecture as an L^2–analogue of the Melvin–Morton conjecture? (Along the line of Gukov's expectation ?)

(2) Whether there is a "volume conjecture with parameter"? Is our invariant $\Delta_K^{(2)}(t)$ related to the volume $Vol(\rho)$ for $\rho : \Gamma \to SL_2(C)$? (A–polynomial of the knot)

See "An L^2-Alexander invariant for knots. Commun. Contemp. Math. 8 (2006), no. 2, 167–187." with W. Zhang.