An Introduction to General and Homogeneous Finsler Geometry

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Outline

1. Introduction: what is Finsler geometry

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2. Minkowski norm and Finsler metric
3. Examples: Riemannian metric, Randers metric, (α, β)-metric, etc.

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I assume the audience understand the fundamental knowledge on smooth manifold and Riemannian geometry, but are curious what is going on in Finsler geometry.

The purpose of this talk is to guide your tour in the fairy land of Finsler geometry, from the most fundamental concepts to the most recent progress.

Starting from Riemannian geometry: a Riemannian metric tells you the length of each tangent vector. In each tangent space, the length of vector is the same as in an Euclidean space. A perfect match (approximation) for the static reality (i.e. the world without forces).

Riemannian metrics can be characterized by its **indicatrix** (the points marking all the unit vectors in each tangent space), which must always be an ellipsoid centered at the origin.

With forces presented, Riemannian metric may not be a good model. For example, with a very strong side wind, the most efficient path walking (sailing) from one place to another is not the strait line (big circle when considering the earth as a standard round disk) between the two points.

If we consider the effect of the forces when we define the lengths of tangent vectors, then the indicatrix may not be an ellipsoid any more, or it could be an ellipsoid but not centered at the origin. Then Finsler geometry appears.

- The concept of Finsler geometry is firstly purposed by Riemann in his talk of 1854. The Riemannian geometry is only a special case.

- In 1900, Hilbert purposed 23 problems, in which two are related to Finsler geometry.

- The name "Finsler geometry" appear after Finsler's thesis in 1918.

- Berwald, Landsberg, etc., established the early framework for Finsler geometry.

Compared to Riemannian geometry, Finsler geometry is not that popular during the most part of the twentieth century because its complexity for notions and calculations.

But nowadays, it is getting more and more important because scientists discover or re-discover its applications in a lot of areas.

S. S. Chern strongly advocated Finsler geometry. Since 1990, there has been a rapid development in Finsler geometry. The representative researchers includes Z. Shen, D. Bao, Y. Shen, and many other Chinese and European geometers. There are many interesting unsolved problems in Finsler geometry.

Let M be a connected n-dimensional smooth manifold. We first define the Finsler metric in each tangent space, which is called a Minkowski norm, which can be more generally defined on each real vector space.

Definition

A **Minkowski norm** on an n-dimensional real vector space $\mathbf{V}$ is a continuous function $F : \mathbf{V} \to [0, +\infty)$ satisfying the following conditions:
(1) Regularity: the restriction of F to $\mathbf{V}\backslash 0$ is a positive smooth function.
(2) Homogeneity: for any $y \in \mathbf{V}$ and $\lambda \geq 0$, $F(\lambda y) = \lambda F(y)$.
(3) Strong Convexity: given any basis $\{e_1, \ldots, e_n\}$, the Hessian matrix

$$(g_{ij}(x, y)) = \left(\frac{1}{2}\frac{\partial^2}{\partial y^i \partial y^j}[F^2(y)]\right)$$

is positive definite when $y = y^i e_i \neq 0$.

Definition

A **Finsler metric** on M is a continuous function $F : TM \to [0, +\infty)$ such that:
(1) The restriction of F to the slit tangent bundle $TM \backslash 0$ is smooth.
(2) The restriction of F to each tangent space is a Minkowski norm.

A Finsler metric F is a Riemannian metric when the Hessian matrix $(g_{ij}(x, y))$ is only dependent to the x-variables, with respect to all standard local coordinate systems. Then it defines a global smooth section $g_{ij}(x)dx^i dx^j$ of $\mathrm{Sym}^2(T^*M)$ which is usually referred to as the Riemannian metric.

Unless otherwise specified, the metrics studied in Finsler geometry are assumed to be non-Riemannian.

A Randers metric is the most simple and the most important (non-Riemannian) Finsler metric, which is of the form $F = \alpha + \beta$, where $\alpha : TM \to \mathbb{R}$ is a Riemannian metric, and $\beta : TM \to \mathbb{R}$ is a one-form. The definition for Minkowski norm and Finsler metric would require the $|\beta(x)|_{\alpha(x)} < 1$ at each point.

All Randers metrics are one-to-one given by the pairs (α, β) satisfying the condition mentioned above.

It is easy to see that a Randers metric is Riemannian iff it is reversible iff $\beta \equiv 0$.

The indicatrix of a Randers metric F is an ellipsoid in each tangent space. Let W be the vector field marking the centers of all the ellipsoid, and h be the Riemannian metric which indicatrix is the the same as that of F except that the centers are moved back to the origin. Obviously $h(W) < 1$ everywhere.

All Randers metrics are one-to-one given by the pairs (h, W) satisfying $h(W) < 1$ everywhere. We call (h, W) the *navigation datum* for the Randers metric F.

The relation between the defining pair (α, β) and the navigation datum (h, W) can be seen from the following formulas:

$$\alpha(x, y) = \frac{\sqrt{h(y, W(x))^2 + h(y^2)(1 - h(W(x), W(x)))}}{1 - h(W(x), W(x))},$$

$$\beta(x, y) = -\frac{h(y, W(x))}{1 - h(W(x), W(x))}.$$

(α, β)-metrics can be seen as a generalization of Randers metrics. They are of the form $F = \alpha\phi(\beta/\alpha)$, where ϕ is a positive smooth function. The convexity requirement implies

$$\phi - s\phi' + (b^2 - s^2)\phi'' > 0$$

for all $|s| \leq b = ||\beta||_\alpha$.

On a Finsler manifold, the arc length can be defined for any piecewise smooth curves. Then a geodesic can be defined as a piecewise smooth curve satisfying the locally minimizing principle. For the convenience we will always parametrize a geodesic $c(t)$ to have positive constant speed, i.e., $F(\dot{c}(t)) = F(\frac{d}{dt}c(t)) = \mathrm{const}$.

Geodesics with positive constant speed can be equivalently defined using the *geodesic spray* vector field on $TM\backslash 0$.

The geodesic spray can be locally presented as $\mathbf{G} = y^i\partial_{x^i} - 2\mathbf{G}^i\partial_{y^i}$ for any standard local coordinate system, where

$$\mathbf{G}^i = \frac{1}{4}g^{il}([F^2]_{x^k y^l}y^k - [F^2]_{x^l}).$$

Notice that it is in fact globally defined.

Then a curve $c(t)$ on M is a geodesic of positive constant speed if and only if $(c(t), \dot{c}(t))$ is an integration curve of **G**. Thus on a standard local coordinates, a geodesic $c(t) = (c^i(t))$ satisfies the equations

$$\ddot{c}^i(t) + 2\mathbf{G}^i(c(t), \dot{c}(t)) = 0, \quad \forall i.$$

Then we discuss the curvature. In Finsler geometry, there are two classes of curvature:

(1) Generalization of the curvature concepts from Riemannian geometry, describing how the space curves.

(2) Geometric quantities which vanish for Riemannian manifolds, describing how non-Riemannian those spaces are.

The most typical Riemannian curvatures: Riemann curvature, flag curvature, Ricci curvature, scalar curvature, etc..

Riemann curvature appears in the Jacobi equation for a variation for a family of constant speed geodesics. For standard local coordinates, it can be presented as a linear operator $R_y = R^i_k(y)dx^k \otimes \partial_{x^i} : T_xM \to T_xM$, where

$$R^i_k(y) = 2\frac{\partial}{\partial x^k}\mathbf{G}^i - y^j\frac{\partial^2}{\partial x^j \partial y^k}\mathbf{G}^i + 2\mathbf{G}^j\frac{\partial^2}{\partial y^j \partial y^k}\mathbf{G}^i - \frac{\partial}{\partial y^j}\mathbf{G}^i\frac{\partial}{\partial y^k}\mathbf{G}^j.$$

Riemann curvature can also be defined from the structure equation for the curvature of the Chern connection. There are several connections in Finsler geometry, Chern connection, Cartan connection, Berwald connection. But no Levi-Civita connection.

The Riemannian curvature $R_y : T_xM \to T_xM$ is self adjoint with respect to $\langle , \rangle_y^F$. Its trace is the Ricci curvature. There are several other ways to define the Ricci curvature in Finsler geometry.

Definition

Gievn $x \in M$, let **P** be a tangent plane in some T_xM, and y a nonzero vector in **P**. Suppose **P** is linearly spanned by y and v. Then the **flag curvature** for the triple $(x, y, \mathbf{P})$ is defined as

$$K^F(x, y, \mathbf{P}) = \frac{\langle R_y^F(v), v\rangle_y^F}{\langle y, y\rangle_y^F \langle v, v\rangle_y^F - (\langle y, v\rangle_y^F)^2}.$$

The flag curvature is a generalization of sectional curvature in Riemannian geometry, i.e. when F is a Riemannian metric, it is just the sectional curvature for $(x, \mathbf{P})$, which is independent of the choice of y.

Some typical non-Riemannian curvatures: Cartan curvature, S-curvature, etc..

The Cartan curvature is given by the third derivative of F^2 for the y-variables, i.e.

$$C_y^F(u, v, w) = \frac{1}{2}\frac{d}{dt}\langle u, v\rangle_{y+tw}^F|_{t=0}$$

$$= \frac{1}{4}\frac{\partial^3}{\partial r \partial t \partial s} F^2(y + ru + sv + tw)|_{r=s=t=0}.$$

Z. Shen defined S-curvature in 1997 (locally described while globally defined):

Firstly, we have the Busemann-Hausdorff volume form $dV_{BH} = \sigma(x)dx^1 \cdots dx^n$, where

$$\sigma(x) = \omega_n/\text{Vol}\{(y^i) \in \mathbb{R}^n | F(x, y^i\partial_{x^i}) < 1\},$$

here Vol denotes the volume of a subset with respect to the standard Euclidian metric on $\mathbb{R}^n$, and $\omega_n = \text{Vol}(B_n(1))$. Secondly, we have distortion function

$$\tau(x,y) = \ln\sqrt{\det(g_{ij}(x,y))} - ln\sigma(x)$$

Finally, the S-curvature is $S(x, y)$ on $TM\backslash 0$ is defined as the derivative of $\tau(x, y)$ in the direction of $\mathbf{G}(x, y)$.

(1) S. S. Chern and D. Bao generalized the Gauss-Bonnet-Chern formula to Finsler manifolds of Landsberg type.

(2) Z. Shen proved a volume comparison theorem in Finsler geometry, and proved the compactness for the space of Finsler metrics with curvature, volume and diameter bounds. S-curvature plays an important role here.

(3) D. Bao, Robles and Z. Shen classified all the Randers spaces of constant flag curvature, using the navigation process.

(4) R. Bryant construct a family of strange Finsler metrics of constant flag curvatures on spheres, using integrable system, Zoll manifold, contact geometry, complex geometry, etc..

The classification of Randers spaces of constant flag curvature, by D. Bao, C. Robles and Z. Shen:

Theorem

A Randers metric F has the constant flag curvature κ iff its navigation datum (h, W) satisfies the following conditions:
(1) The metric h has constant curvature $\kappa + \frac{1}{4}\mu^2$ for some constant μ.
(2) The vector field W is μ-homothetic for h, i.e. $\mathcal{L}_W h = 2\mu h$, where $\mathcal{L}$ is the Lie derivative.

The importance of Randers spheres of constant flag curvature:

- The discovery of spheres with only finite closed geodesics (Katok metric)

- Anosov Conjecture is modelled on Randers spheres of constant flag curvature.

- A lot of work on Anosov Conjeture and related topics by Y. Long, V. Bangert, H.B. Rademacher, H. Duan, W. Wang, etc.

- Recent study on geodesic behavior of S^2 with constant flag curvature.

Generally speaking, the complexity of the calculations in Finsler geometry scare people away. If we impose the isometric symmetry of a Lie group action, then this complexity can be reduced. When a connected Lie group acts transitively and isometrically on a Finsler space (M, F), we call (M, F) a *homogeneous Finsler space*, and M can be presented as $M = G/H$.

The basic algebraic setup for a homogeneous Finsler space $M = G/H$ is the following:
(1) Denote $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{h} = \mathrm{Lie}(H)$, and we can find an $\mathrm{Ad}(H)$-invariant decomposition $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$.

(2) We can identify $\mathfrak{m}$ with the tangent space $T_{eH}(G/H)$. The homogeneous metric F is one-to-one determined by an $\mathrm{Ad}(H)$-invariant Minkowski norm on $\mathfrak{m}$.

(3) We only need to study the curvature of M at eH, which are much simpler.

(4) We can calculate something in homogeneous Finsler geometry, and study the connection between the geometric properties and the algebraic conditions.

An example is the following theorem:

Theorem

Let $(G/H, F)$ be a connected homogeneous Finsler space, and $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ be an $\mathrm{Ad}(H)$-invariant decomposition for G/H. Then for any linearly independent commuting pair u and v in $\mathfrak{m}$ satisfying $\langle [u, \mathfrak{m}], u\rangle_u^F = 0$, we have

$$K^F(o, u, u\wedge v) = \langle U(u, v), U(u, v)\rangle_u^F / (\langle u, u\rangle_u^F \langle v, v\rangle_u^F - [\langle u, v\rangle_u^F]^2),$$

where $U(u, v) \in \mathfrak{m}$ satisfy
$$\langle U(u, v), w\rangle_u^F = \frac{1}{2}(\langle [w, u]_{\mathfrak{m}}, v\rangle_u^F + \langle [w, v]_{\mathfrak{m}}, u\rangle_u^F), \quad \forall w \in \mathfrak{m}.$$

L. Huang has more general formula for the flag curvature of a homogeneous Finsler space. This theorem can be viewed as a corollary of L. Huang's formula. But L. Huang's formula is much more complicated. The most useful case of L. Huang's formula is the one given above.

Using this theorem, we made a big progress in the classification for homogeneous Finsler spaces with positive flag curvature.

(1) Clifford-Wolf translation, Clifford-Wolf homogeneity and Killing vector field of constant length in Finsler geometry (some progress).

(2) Classify homogeneous Finsler spaces with positive flag curvature (some big progress, but not finished).

(3) Construct the theoretical framework for piecewise flat Finsler geometry (a combinatoric method in studying Finsler geometry, a new way just started).

Two popular topics in Finsler geometry:

(1) Finsler sphere of constant flag curvature $K \equiv 1$.

(2) Number of prime closed geodesics.

Recently, R. Bryant, P. Foulon, S. Ivanov, V. Matveev and W. Ziller used Lie theory (and integrable system) to describe the behavior of geodesics in a Finsler S^2 with $K \equiv 1$.

I generalize their method to higher dimensions, and proved the following theorems recently (see my most recent three preprints in arXiv):

Theorem

Let $(M, F) = (S^n, F)$ with $n > 1$, $K \equiv 1$ and only finite prime closed geodesics. Then $I_o(M, F)$ is a torus and

$$0 < \dim I(M, F) \leq [\frac{n+1}{2}].$$

Theorem

Let $(M, F) = (S^n, F)$ with $n > 1$, $K \equiv 1$ and only finite prime closed geodesics. Assume $m = \dim I(M, F)$, then there exists at least m geometrically distinct reversible closed geodesics, with non-trivial actions of $I_o(M, F)$.

Theorem

Let $(M, F) = (S^n, F)$ *with* $n > 1$, $K \equiv 1$ *and only finite prime closed geodesics. Assume* $\dim I(M, F) = [\frac{n+1}{2}]$, *then all closed geodesics are reversible and there exists exactly* $[\frac{n+1}{2}]$ *reversible prime closed geodesics, i.e. the estimate in the Anosov conjecture is valid and sharp in this case.*

Theorem

Let $(M, F) = (S^n, F)$ be a Finsler sphere with $n > 1$, $K \equiv 1$ and only finite orbits of prime closed geodesics. Assume the group H of isometries preserving each closed geodesics has a dimension m, then there exists m geometrically distinct orbits $\mathcal{B}_i$'s of prime closed geodesics, such that for each i, the union B_i of the geodesics in $\mathcal{B}_i$ is a totally geodesic submanifold with nontrivial H_0-action.

Theorem

A homogeneous Finsler metric on a sphere S^n with $n > 1$ is a geodesic orbit metric, i.e. each geodesic is the orbit of a one-parameter subgroup, iff its connected isometry group is not $Sp(k)$ for some $k > 0$.

Theorem

A homogeneous Finsler metric F on a sphere S^n with $n > 1$, $K \equiv 1$ is a geodesic orbit metric, iff it is Randers.

Theorem

A homogeneous Finsler sphere of constant flag curvature must be Randers if its connected isometry group is not $Sp(k)$ for some k.

or equivalently

Theorem

A homogeneous Finsler sphere which is not of the type $Sp(k)/Sp(k-1)$ has constant flag curvature only if it is Randers.

This is the end.

Sincerely thank Wenchuan Hu for inviting me, and the audience for your patience.

Thank you very much!