Higher Nahm Transform

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Introduction

Let X be a four manifold., and $E \to X$ be a bundle with a connection:

$$A : C^\infty(E) \to C^\infty(E \otimes T^*X) = C^\infty(E) \otimes_{C^\infty(X)} C^\infty(T^*X)$$

Non commutative framework:

$$\begin{aligned} \mathcal{E}_0 : &\quad \text{finitely generated projective } \mathcal{A} \text{ module,} \\ \mathcal{A} : &\quad * - \text{algebra,} \\ \Omega^1 : &\quad \mathcal{A} \text{ bimodule.} \end{aligned}$$

Non commutative connection, or q-connection is a linear map:

$$\nabla : \mathcal{E}_0 \to \mathcal{E}_0 \otimes_{\mathcal{A}} \Omega^1$$

Basic questions:

(1) Find 'nice' connections, and

(2) Find a systematic way to construct q-connections.

Hints:

(1) Yang-Mills theory, and

(2) Nahm transform.

Instanton

Let X be a Riemannian compact oriented smooth four manifold, and $E \to X$ be an $SU(2)$ vector bundle. E is determined by $c_2(E) \in \mathbb{Z}$.
Choose a connection A.

Definition: The Yang-Mills functional:

$$\text{YM}(A) := \int_X ||F_A||^2 \text{ vol}.$$

For fixed $E, (X, g)$, this gives a linear map:

$$\text{YM} : \{ \text{ connections } \} \to [0, \infty)$$

Q: What are the critical values of this ?

Let $(V, < >)$ be a 4-dimensional Euclidean plane. The exterior algebra $\wedge^* V$ admits an involution:

$$* : \ \wedge^2 V \cong \wedge^2 V$$

with $*^2 = 1$, where $*$ is the Hodge $*$. So it admits an orthogonal decomposition:

$$\wedge^2 V = \wedge^+ V \oplus \wedge^- V$$

with respect to ± 1 eigenspaces.

Topological lower bound

Since X is 4-dimensional, $\wedge^2 X = \wedge^2 T^* X$ splits as:

$$\wedge^2 X = \wedge^+ X \oplus \wedge^- X.$$

Let A be a connection and F_A be the curvature. Then:

$$F_A = F_A^+ \oplus F_A^-.$$

$$\begin{aligned} \mathrm{YM}(A) &= \int_X (||F_A^+||^2 + ||F_A^-||^2) \text{ vol} \\ &\geq \int_X (-||F_A^+||^2 + ||F_A^-||^2) \text{ vol} \\ &= 8\pi^2 c_2(E). \end{aligned}$$

$$\text{Equality} \iff F_A^+ = 0$$

ASD connection attains the critical value !

Nahm Transform

Suppose X is spin, and $S^{\pm} \to X$ be the spinor bundle with the Dirac operator:

$$D : L^2(S^+) \to L^2(S^-).$$

For each (E, A), Dirac operator with the coefficient:

$$D_A : L^2(S^+ \otimes E) \to L^2(S^- \otimes E)$$

is associated.
For $\rho \in \hat{T}$, one can twist :

$$E_\rho := E \otimes L_\rho$$

by the flat line bundle:

$$L_\rho = \tilde{X} \times_\rho \mathbb{C}.$$

Consider:

$$D_A^\rho : L^2(S^+ \otimes E_\rho) \to L^2(S^- \otimes E_\rho).$$

The Weitzenböck formula:

$$(D_A^\rho)^* D_A^\rho = \nabla_A^* \nabla_A + \frac{Sc}{4} - F_A^+.$$

Suppose (1) $\nabla_A(u) = 0$ implies $u = 0$ and (2) $Sc \geq 0$. Then:

$$\ker D_A^\rho = 0$$

if A is ASD. This gives a vector bundle:

$$\check{E}_A := \coprod_{\rho \in \hat{T}^4} \text{Coker } D_A^\rho \to \hat{T}^4$$

$$\bigcap$$

$$\mathcal{H} := \coprod_{\rho \in \hat{T}^4} L^2(S^- \otimes E_\rho) \quad \text{flat Hilbert bundle.}$$

$\mathcal{H}$ is equipped with a flat connection ∇. Let:

$$P_A : \mathcal{H} \to \check{E}$$

be the orthogonal projection. Then we obtain the induced connection on $\check{E}$:

$$\check{A} := P_A \circ \nabla.$$

Theorem: Let $X = T^4$ be the flat torus.
If A is ASD, then $\check{A}$ is also ASD.

Spectral triple

Let $(\mathcal{A}, H, D)$ be a spectral triple so that:

(1) there is a $*$ representation $\pi : \mathcal{A} \to \mathcal{L}(H)$,

(2) $[D, a] \in \mathcal{L}(H)$ for any $a \in \mathcal{A}$, and

(3) $a(1 + D^2)^{-1} \in K(H)$ is compact for all $a \in \mathcal{A}$.

Example: Let X be spin. Then

$$(\mathcal{A}, H, D) = (C^\infty(X), L^2(X; S), D)$$

where D in the r.h.s. is the Dirac operator.
We call it as the **spin spectral triple**.

Quantised calculus

Let $(\mathcal{A}, H, D)$ be a spectral triple, and assume $\ker D = 0$. Put $F := \frac{D}{|D|}$, which is order 0 in the case of the spin spectral triple.

Definition: For $a \in \mathcal{A}$,

$$\hat{d}(a) := i[F, a] \in \mathcal{K}(H),$$
$$\Omega_D^* := \text{ span } \{ \ a^0 \hat{d} a^1 \dots \hat{d} a^q \ \}.$$

$$\hat{d} : \Omega_D^q \to \Omega_D^{q+1}.$$

Lemma:(Connes) Let $(\mathcal{A}, H, D)$ be a spin spectral triple. Then for $* \leq 2$, there is a bimodule map:

$$c : \Omega_D^* \to T^* \wedge^1 X$$

which sends $\hat{d}a \to da$, where the r.h.s. is the tensor product fields of one forms.

Q-connection

Let $(\mathcal{A}, H, D)$ be a spectral triple, and $\mathcal{E}_0$ be a finitely generated projective module.

Definition: A linear map:

$$\nabla : \mathcal{E}_0 \to \mathcal{E}_0 \otimes_{\mathcal{A}} \Omega^1_D$$

is a q-connection, if it satisfies the Leibnitz rule:

$$\nabla(e \otimes a) = \nabla(e)a + e \otimes \hat{d}a.$$

Let:

$$\Theta := \nabla^2 : \mathcal{E}_0 \to \mathcal{E}_0 \otimes_{\mathcal{A}} \Omega^2_D$$

be the curvature. It is an $\mathcal{A}$ module map.

Dixmier trace

Let $B \in \mathcal{K}_+(\mathcal{H})$ with eigenvalues:

$$\lambda_1 \geq \lambda_2 \geq \cdots > 0$$

$$||B||_{(1,\infty)} := \sup_N \log(1+N)^{-1} \sum_{i=1}^{N} \lambda_i.$$

Definition: Dixmier trace:

$$\mathrm{Tr}_\omega : \mathcal{L}^{(1,\infty)}(\mathcal{H})_+ \to [0,\infty),$$

$$\mathrm{Tr}_\omega(B) = \omega - \lim_N \log(1+N)^{-1} \sum_{i=1}^{N} \lambda_i.$$

Connes trace formula

Theorem: Let X be n dimensional and $P \in \Psi^{-n}(X, E)$. Then:

$$P \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$$

and:

$$\mathrm{Tr}_\omega(P) = \frac{1}{n} \text{ Res } (P)$$

where the right hand side is the Wosdizcki residue.

Connes-Yang-Mills functional

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple, and $\mathcal{E}_0$ be a finitely generated projective $\mathcal{A}$ module.
q-connection $\nabla : \mathcal{E}_0 \to \mathcal{E}_0 \otimes_{\mathcal{A}} \Omega^1_D$, and
its curvature $\Theta := \nabla^2 : \mathcal{E}_0 \to \mathcal{E}_0 \otimes_{\mathcal{A}} \Omega^2_D$ is an $\mathcal{A}$ module.

Hence it induces an endomorphism:

$$\Theta : \mathcal{E}_0 \otimes_{\mathcal{A}} \mathcal{H} \to \mathcal{E}_0 \otimes_{\mathcal{A}} \mathcal{H}$$

Definition: pre-Connes-Yang-Mills functional is given by:

$$I(\nabla) := \mathrm{Tr}_\omega(\Theta^2).$$

Let $\mathcal{E}_0 = C^\infty(X, E)$ with spin spectral triple. Then ∇ induces a classical connection:

$$\nabla \to \nabla_c := (1 \otimes c) \circ \nabla : \ \mathcal{E}_0 \to \mathcal{E}_0 \otimes_{\mathcal{A}} \Omega^1(X).$$

Theorem: Let A be a classical connection. Then:

$$\frac{1}{16\pi^2}\mathrm{YM}(A) = \inf_{\nabla_c = A} \ I(\nabla).$$

The right hand side is the Connes-Yang-Mills functional

$$\mathrm{CYM}(A).$$

Notice that formulation of the CYM requires space.

Dixmier Γ-trace

Let $\mathcal{E}$ be a Hilbert $C_r^*(\Gamma)$ module, and

$$\mathrm{tr}_\Gamma : \mathcal{K}(\mathcal{E}) \to \mathbb{C}$$

be the composition of von Neumann trace with the standard trace.

Generalised singular value function:

$$\mu(T) : [0, \infty) \to [0, \infty)$$

for $T \in K(\mathcal{E})_+$ is given by:

$$\mu(T)_t = \inf\{ \ s \geq 0 : \mathrm{tr}_\Gamma(\chi_{(s,\infty)}(T)) \leq t \ \}.$$

Let:

$$||T|| := \sup_{t>0} \log(1+t)^{-1} \int_0^t \mu_s(T) ds,$$
$$\mathcal{L}^{(1,\infty)}(\mathcal{E}) := \{ T \in \mathcal{K}(\mathcal{E}) : ||T|| < \infty \}.$$

Definition: Dixmier Γ-trace is given by:

$$\mathrm{Tr}_\omega^\Gamma(A) := \omega - \lim_{t\to\infty} \log(1+t)^{-1} \int_0^t \mu_s(A) ds,$$

for $A \in \mathcal{L}^{(1,\infty)}(\mathcal{E})$.

Higher Connes-Yang-Mills functional

Let $E \to X$ be an $SU(2)$ bundle and X be spin with $\Gamma = \pi_1(X)$.
Let $(\mathcal{A}, \mathcal{H}, D)$ be the spin spectral triple.
For $\gamma = \tilde{X} \times_\Gamma C_r^*(\Gamma)$, let:

$$\mathcal{E}_0 = C^\infty(X; E \otimes \gamma)$$

For a connection A_c on E, let $\mathbb{A}_c$ be the induced connection on $E \otimes \gamma$. Consider their curvatures:

$$\begin{aligned} A_c &\Rightarrow \theta \qquad \text{on } E, \\ \mathbb{A}_c &\Rightarrow \Theta \qquad \text{on } E \otimes \gamma. \end{aligned}$$

Recall for a q-connection A, one can associate the classical connection:

$$A_c = 1 \otimes c \circ A : \mathcal{E}_0 \to \mathcal{E}_0 \otimes_{\mathcal{A}} \Omega^1(X).$$

Definition:

The higher Connes-Yang-Mills functional is given by:

$$\mathrm{HCYM}(\mathbb{A}) := \inf_{c(\mathbb{A})=\mathbb{A}_c} \mathrm{Tr}^{\Gamma}_{\omega}(\Theta^2).$$

Theorem(K-S-W):

Let $X = T^4$ be the standard torus. Then the equality holds:

$$\mathrm{CYM}(A_c) = \mathrm{HCYM}(\mathbb{A}_c).$$

In particular if A_c is ASD, then $\mathbb{A}_c$ minimizes the higher CYM functional.

Higher Nahm transform

Let $S \to X$ be the spinor bundle and $E \to X$ be an $SU(2)$ bundle with a connection A_c. Consider the Dirac operator with coefficient:

$$D_{A_c} : L^2(X; S_E^+ \otimes \gamma) \to L^2(X; S_E^- \otimes \gamma).$$

Suppose $\ker D_{A_c} = 0$. Then:

$$\mathcal{E}_{A_c} := \text{ coker } D_{A_c} \quad \subset \quad L^2(X; S_E^- \otimes \gamma)$$

is a finitely generated projective $C_r^*(\Gamma)$ module.

Recall Kasparov's stabilization:

$$L^2(X; S_E^- \otimes \gamma) \oplus (H \otimes C_r^*(\Gamma)) \;\cong\; H \otimes C_r^*(\Gamma)$$

as Hilbert $C_r^*(\Gamma)$ modules. Then there is a module projection:

$$P_{A_c} : H \otimes C_r^*(\Gamma) \to \mathcal{E}_{A_c}$$

Let $\mathbb{C}\Gamma \subset C^\infty(\Gamma) \subset C_r^*(\Gamma)$ be a smooth algebra. Choose and fix a spectral triple:

$$(\mathcal{A} = C^\infty(\Gamma), H, D)$$

There is the trivial q-connection $\hat{d}$ on $H \otimes C_r^*(\Gamma)$. Composition with the projection produces a q-connection:

$$\hat{d}_{A_c} := P_{A_c} \circ \hat{d} \qquad \text{over } \mathcal{E}_{A_c}$$

Definition: The higher Nahm transform is an assignment:

$$(E, A) \to (\mathcal{E}_A, \hat{d}_A)$$

Let $X = T^4$. Then $\hat{d}_A$ minimizes the Connes-Yang-Mills functional over Pic $(\mathbb{Z}^4)$, if A is ASD.

Corollary: Let $X = T^4$ be the flat torus. The higher Nahm transform sends the minimizer of the higher Connes-Yang-Mills functional to the minimizer of the Connes-Yang-Mills functional.