Reconstructing the Bost–Connes semigroup actions from K-theory

Yosuke KUBOTA joint work with Takuya Takeishi

iTHEMS, RIKEN

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Bost-Connes system

$$\begin{array}{ccc} \text{number field} & \to & \text{semigroup action} & \to & \mathsf{C*-dynamical system} \\ & \mathcal{K} & \to & \mathcal{K} & \sim \\ & \mathcal{K} & \sim & \mathcal{K} & \mathcal{K} \end{array}$$

Arithmetics of K is reflected to the dynamics of (A_K, σ_K)

• The partition function
$$= \zeta_{\kappa}$$
.

3 For
$$\beta \leq 1$$
, $|\text{KMS}_{\beta}(A_{\kappa}, \sigma_{\kappa})| = 1$.

- For $\beta > 1$, $G_{K}^{ab} \frown exKMS_{\beta}(A_{K}, \sigma_{K})$ is free and transitive.
- So There is a K-subalgebra A₀ ⊂ A such that φ(A₀) = K^{ab} for φ ∈ KMS_β(A_K, σ_K).

(Bost–Connes '95, Connes–Marcolli–Ramachandran'05, Ha–Paugam'05, Laca–Larsen–Neshveyev'09, Yalkinoglu'13)

Classification of (A_K, σ_K) : motivation

Neukirch–Uchida theorem

If $G_K \cong G_L$, then $K \cong L$.

However, $G_{\mathcal{K}}^{\mathrm{ab}} \cong G_{\mathcal{L}}^{\mathrm{ab}}$ does not imply $\mathcal{K} \cong \mathcal{L}$.

Question(Cornelissen–Marcolli '11)

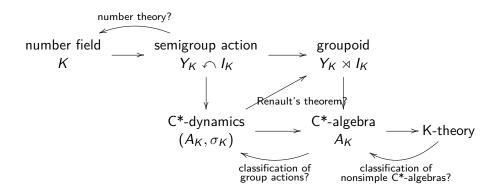
 $(A_K, \sigma_K) \cong (A_L, \sigma_L) \Rightarrow K \cong L?$

- $(A_K, \sigma_K) \cong (A_L, \sigma_L) \Rightarrow \zeta_K = \zeta_L.$ (If both K and L are Galois extensions, then $\zeta_K = \zeta_L \Rightarrow K = L$).
- (Takeishi'16) If $A_K \cong A_L$, then $\zeta_K = \zeta_L$ and $h_K^1 = h_L^1$.

Another question

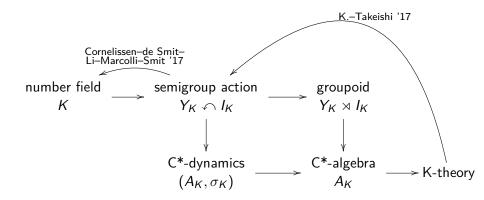
$$A_K \cong A_L \Leftrightarrow \zeta_K \cong \zeta_L \text{ and } h_K^1 = h_L^1?$$

At first we expected...



- $A_{\mathcal{K}} = C(Y_{\mathcal{K}}) \rtimes I_{\mathcal{K}} \curvearrowleft \mathbb{T}^{\mathcal{P}_{\mathcal{K}}}$: dual action. $\sigma_{\mathcal{K}}$ is its restriction to a subgroup $\mathbb{R} \subset \mathbb{T}^{\mathcal{P}_{\mathcal{K}}}$.
- Renault's theorem: if $C^*(\mathcal{G}_K) \cong C^*(\mathcal{G}_L)$ induces $C(\mathcal{G}_K^0) \cong C(\mathcal{G}_L^0)$, then $\mathcal{G}_K \cong \mathcal{G}_L$.

What we actually did



Main theorem

Theorem (K.-Takeishi '17)

The following are equivalent:

- $K \cong L$ as fields,

•
$$(A_K, \sigma_K) \cong (A_L, \sigma_L)$$
 as \mathbb{R} -C*-algebras,

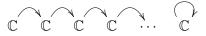
•
$$A_K$$
 and A_L are $KK(X)$ -equivalent $+\alpha$,

• A_K and A_L have the same "K-theoretic data".

What we actually show is $(7) \Rightarrow (2)$.

Toeplitz algebra and generalizations

The Toeplitz algebra *T* = *C*^{*}(*s*) ⊂ B(*l*²N) is written as the semigroup crossed product *C*(N⁺) ⋊ N.



• For $A \curvearrowleft \mathbb{Z}$, let $\mathcal{T}(A, \alpha) = (\mathcal{C}(\mathbb{N}^+) \otimes A) \rtimes \mathbb{N}$.

$$A A A A A \cdots A$$

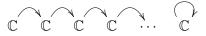
- Then $0 \to A \otimes \mathbb{K} \to \mathcal{T}(A, \alpha) \to A \rtimes \mathbb{Z} \to 0$ is exact.
- If we have $A \to B$, there is a semigroup crossed product $\mathcal{T}(A, \alpha, B)$

$$B \xrightarrow{B} B \xrightarrow{B} B \xrightarrow{\cdots} A$$

such that $0 \to B \otimes \mathbb{K} \to \mathcal{T}(A, \alpha, B) \to A \rtimes \mathbb{Z} \to 0$ is exact.

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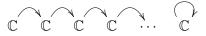
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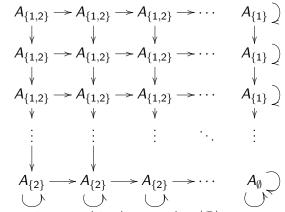
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Multi-Toeplitz dynamical system

If we have a compatible family of C*-algebras $A_F \curvearrowleft \mathbb{Z}^{\mathcal{P} \setminus F}$ for $F \subset \mathcal{P}$ and $\mathbb{Z}^{\mathcal{P} \setminus F'}$ -*-homomorphisms $A_F \to A_{F'}$ for $F \subset F'$, then we have a $\mathbb{N}^{\mathcal{P}}$ -action on a continuous field of C*-algebras $\mathcal{A} = \bigsqcup A_F$.



In particular we are interested in the case that $|\mathcal{P}| = \infty$.

- A $\mathbb{N}^{\mathcal{P}}$ -invariant open subset U of $(\mathbb{N}^+)^{\mathcal{P}}$ corresponds to an ideal $\mathfrak{A}(U)$.
- $(\mathbb{N}^{\mathcal{P}}\text{-invariant open subsets of } (\mathbb{N}^+)^{\mathcal{P}}) \leftrightarrow (\text{open subsets of } 2^{\mathcal{P}}),$ where 2 is equipped with the topology $\{\emptyset, \{0\}, 2\}.$
- An locally closed subset (i.e. a subset of the form Z = U \ V) of 2^P corresponds to a subquotient 𝔅(Z).
- When *P* is finite, {*F*} is locally closed and 𝔅({*F*}) ≅ *A_F* ⋊ ℤ^{P\F} ⊗ 𝔅. When *P* is infinite, {*F*} is locally closed iff *F* is finite.
- By continuity, we have $A_S = \varinjlim_{F \subset CS} A_F$. Hence a multi-Toeplitz dynamical system is reconstructed from the family $\{A_F\}_{F \subset CP}$.
- If we want to reconstruct $\mathcal{A} \curvearrowright \mathbb{N}^{\mathcal{P}}$ from \mathfrak{A} , the remaining task is to reconstruct $\mathcal{A}_F \curvearrowright \mathbb{Z}^{\mathcal{P} \setminus F}$ from $\mathfrak{A}(\{F\}) \sim \mathcal{A}_F \rtimes \mathbb{Z}^{\mathcal{P} \setminus F}$.

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Algebraic number theory

- K: number field i.e. $[K : \mathbb{Q}] < \infty$
- $\mathcal{O}_{\mathcal{K}}$: the ring of integers
- $\mathcal{P}_{\mathcal{K}}$: set of prime ideals of $\mathcal{O}_{\mathcal{K}}$
- $I_{\mathcal{K}} = \{ \text{ideals of } \mathcal{O}_{\mathcal{K}} \} \cong \bigoplus_{\mathcal{P}_{\mathcal{K}}} \mathfrak{p}^{\mathbb{N}} \text{ (prime ideal factorization).}$

•
$$J_{\mathcal{K}} := \bigoplus \mathfrak{p}^{\mathbb{Z}}$$

- $K_{\mathfrak{p}}$: local field (e.g. \mathbb{Q}_{p}), $\mathcal{O}_{\mathfrak{p}}$: ring of integers (e.g. \mathbb{Z}_{p})
- valuation: $v_{\mathfrak{p}} \colon \mathcal{K}_{\mathfrak{p}} \to \mathbb{Z} \cup \{+\infty\}, \ v_{\mathfrak{p}}^{-1}(1) = \mathcal{O}_{\mathfrak{p}}^*, \ \mathcal{K}_{\mathfrak{p}}^* \cong \mathcal{O}_{\mathfrak{p}}^* \times \mathbb{Z}.$
- $\hat{\mathcal{O}}_{K}^{*} := \prod \mathcal{O}_{\mathfrak{p}}^{*}$ and $\mathbb{A}_{K,f}^{*} := \prod'(\mathcal{K}_{\mathfrak{p}}^{*}, \mathcal{O}_{\mathfrak{p}}^{*})$. That is,

$$1 o \hat{\mathcal{O}}_{K}^{*} o \mathbb{A}_{K,f}^{*} o J_{K} o 1$$

• Class field theory: The Artin reciprocity map $\phi \colon \mathbb{A}^*_{K,f} \to G^{ab}_K$ induces an isomorphism $\mathbb{A}^*_{K,f}/\overline{K^*} \cong G^{ab}_K$.

The space Y_K

•
$$0 o \hat{\mathcal{O}}^*_K o \mathbb{A}^*_{K,f} o J_K o 1$$
 restricts to

$$0
ightarrow \hat{\mathcal{O}}_{\mathcal{K}}^*
ightarrow \hat{\mathcal{O}}_{\mathcal{K}}^{\natural}
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ightarrow 0$$

Consider actions Ô^𝔅_K ∩ Ô_K and φ: Ô^𝔅_K → G^{ab}_K
Set Y_K := (Ô_K × G^{ab}_K)/Ô[∗]_K ∩ I_K.

Consider the semigroup action $C(Y_K) \curvearrowleft I_K$.

- $\prod v_{\mathfrak{p}} \colon \hat{\mathcal{O}}_{\mathcal{K}} \to (\mathbb{N}^+)^{\mathcal{P}_{\mathcal{K}}}$ induces the map val: $Y_{\mathcal{K}} \to (\mathbb{N}^+)^{\mathcal{P}_{\mathcal{K}}}$
- Then $C(Y_K) \curvearrowleft I_K$ is a multi-Toeplitz dynamical system with the fibers $C(G_K^F)$, where $G_K^F := G_K^{ab} / \phi(\prod_{\mathfrak{p} \in F} \mathcal{O}_{\mathfrak{p}}^*)$.

Let $A_K := C(Y_K) \rtimes I_K$.

- For finite $F \subset \mathcal{P}_K$, we have a subquotient $A_K(\{F\})$.
- Let $J_K^F := \bigoplus_{p \notin F} \mathbb{Z}^p$. Then $A_K(\{F\}) \cong C(G_K^F) \rtimes J_K^F \otimes \mathbb{K}$.
- According to class fiend theory, G^F_K is a pro-N_F group for some finite subset N_F ⊂ P_Q.

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- According to class fiend theory, G^F_K is a pro-N_F group for some finite subset N_F ⊂ P_Q.

The "K-theoretic data"

In summary, the Bost–Connes C*-algebra A_K is the semigroup crossed product of a multi-Toeplitz dynamical system. Hence we get a subquotient

$$B_K^F := A_K(\{F\}) \sim C(G_K^F) \rtimes J_K^F$$

for each finite subset $F \subset \mathcal{P}_{\mathcal{K}}$. Let $F_{\mathfrak{p}} = F \cup \{\mathfrak{p}\}$. Then

$$0 \to B_{K}^{F_{\mathfrak{p}}} \to A_{K}(\{F,F_{\mathfrak{p}}\}) \to B_{K}^{F} \to 0.$$

Definition

The K-theoretic data of $A_{\mathcal{K}}$ is $\{K_*(B^{\mathcal{F}}_{\mathcal{K}})\}$ and $\{\partial \colon K_*(B^{\mathcal{F}}_{\mathcal{K}}) \to K_{*+1}(B^{\mathcal{F}_p}_{\mathcal{K}})\}$.

Proposition

If
$$\varphi \colon A_K \cong A_L$$
, then there is $\chi \colon \mathcal{P}_K \cong \mathcal{P}_L$ such that φ induces $B_K^F \cong B_L^{\chi(F)}$.

Yosuke KUBOTA (RIKEN)

Reconstruction of profinite completions

Theorem (K.-Takeishi'17)

Let Γ be a free abelian group and let $\varphi \colon \Gamma \to G$ be its pro- \mathcal{N} completion. Then, φ is reconstructed from the inclusion $K_*(C_r^*\Gamma) \to K_*(C(G) \rtimes \Gamma)$.

Corollary

Let Γ_1 and Γ_2 be free abelian groups and let $\varphi_i \colon \Gamma_i \to G_i$ are pro- \mathcal{N} completions. If we have $f \colon \Gamma_1 \cong \Gamma_2$ and

$$\begin{array}{c} \mathrm{K}_{*}(C^{*}\Gamma_{1}) \longrightarrow \mathrm{K}_{*}(C^{*}\Gamma_{2}) \\ \downarrow \\ \mathrm{K}_{*}(C(G) \rtimes \Gamma) \longrightarrow \mathrm{K}_{*}(C(G) \rtimes \Gamma) \end{array}$$

then there is an isomorphism $F: G_1 \to G_2$ such that $F \circ \varphi_1 = \varphi_2 \circ f$.

Example 1: $C(\mathbb{Z}_p) \rtimes \mathbb{Z}$

Let $\Gamma = \mathbb{Z}$, $G = \mathbb{Z}_p$ and $\varphi(1) = 1$. Then, $C(G) = \varinjlim C(G_n)$, where $G_n = \mathbb{Z}/p^n\mathbb{Z}$.

$$\begin{split} \mathrm{K}_0(C(G_n) \rtimes \Gamma) &\cong \mathbb{Z}, \\ \mathrm{K}_1(C(G_n) \rtimes \Gamma) &\cong \mathbb{Z}. \end{split}$$

The map $G_{n+1} \rightarrow G_n$ induces

$$p: \mathrm{K}_0(\mathcal{C}(\mathcal{G}_n) \rtimes \Gamma) \to \mathrm{K}_0(\mathcal{C}(\mathcal{G}_{n+1}) \rtimes \Gamma)$$

1: $\mathrm{K}_1(\mathcal{C}(\mathcal{G}_n) \rtimes \Gamma) \to \mathrm{K}_1(\mathcal{C}(\mathcal{G}_{n+1}) \rtimes \Gamma)$

Therefore, $\mathrm{K}_0(\mathcal{C}(\mathbb{Z}_p) \rtimes \mathbb{Z}) \cong \mathbb{Z}[p^{-1}]$ and $\mathrm{K}_1(\mathcal{C}(\mathbb{Z}_p) \rtimes \mathbb{Z}) \cong \mathbb{Z}$.

Example 2: $C(\mathbb{Z}_p) \rtimes \mathbb{Z}^2$ Let $\Gamma = \mathbb{Z}^2$, $G = \mathbb{Z}_p$ and $\varphi(1,0) = 1$ and $\varphi(0,1) = 1 + p + p^2 + \dots$ Then, $C(G) = \varinjlim C(G_n)$, where $G_n = \mathbb{Z}/p^n\mathbb{Z}$. $K_0(C(G_n) \rtimes \Gamma) \cong \mathbb{Z} \oplus \mathbb{Z}$, $K_1(C(G_n) \rtimes \Gamma) \cong \mathbb{Z} \oplus \mathbb{Z}$.

The map $G_{n+1} \rightarrow G_n$ induces

$$\operatorname{diag}(p,1) \colon \operatorname{K}_0(C(G_n) \rtimes \Gamma) \to \operatorname{K}_0(C(G_{n+1}) \rtimes \Gamma)$$

 $\begin{pmatrix} p & -1 \\ 0 & 1 \end{pmatrix} \colon \operatorname{K}_1(C(G_n) \rtimes \Gamma) \to \operatorname{K}_1(C(G_{n+1}) \rtimes \Gamma)$

Therefore,

$$\begin{split} \mathrm{K}_{0}(\mathcal{C}(\mathbb{Z}_{p}) \rtimes \mathbb{Z}^{2}) &\cong \mathbb{Z}[p^{-1}] \oplus \mathbb{Z}, \\ \mathrm{K}_{1}(\mathcal{C}(\mathbb{Z}_{p}) \rtimes \mathbb{Z}^{2}) &\cong \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbb{Z} + \bigcup \begin{pmatrix} \sum_{j=1}^{n-1} p^{-j} \\ -p^{-n} \end{pmatrix} \mathbb{Z} \subset \mathbb{Z}[p^{-1}]^{2} \end{split}$$

Here, $\operatorname{ker}(\Gamma \to G_n) = \operatorname{K}_1(C^*\mathbb{Z}^2) \cap p^n \operatorname{K}_1(C(\mathbb{Z}_p) \rtimes \mathbb{Z}^2).$

Sketch of the proof

- The group $K_*(C^*\Gamma)$ is isomorphic to $\bigwedge^*\Gamma$.
- For a finitely generated subgroup $\Pi = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k$, let $\beta_{\Pi} := \beta_{v_1} \wedge \cdots \wedge \beta_{v_k}$.
- Then,

$$\delta(\beta_{\mathsf{\Pi}}) := \max\{n \in \mathbb{Z}_{>0} \mid n^{-1}\beta_{\mathsf{\Pi}} \in \mathrm{K}_*(\mathcal{C}(\mathcal{G}) \rtimes \mathsf{\Gamma})\}$$

coincides with $|G/\overline{\varphi(\Pi)}|$.

- By using the data $\delta(\beta_{\Pi})$, we can reconstruct the subgroup $\ker(\Gamma \to G_n) \subset \Gamma$ as an inclusion of subgroups of $\mathrm{K}_*(\mathcal{C}(G) \rtimes \Gamma)$.
- General pro- \mathcal{N} group G is isomorphic to $G_{p_1} \times \cdots \times G_{p_k}$, where G_{p_i} is a pro- p_i group and $\mathcal{N} = \{p_1, \ldots, p_k\}$.
- If we have isomorphisms K_{*}(B^F_K) ≃ K_{*}(B^{χ(F)}_L) compatible with ∂, then is must be compatible with χ: K_{*}(C^{*}J_K) ≃ K_{*}(C^{*}J_L).