Reconstructing the Bost–Connes semigroup actions from K-theory

Yosuke KUBOTA
joint work with Takuya Takeishi

iTHEMS, RIKEN

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Bost–Connes system

number field K $\to$ semigroup action $Y_K \curvearrowleft I_K$ $\to$ C*-dynamical system $A_K \curvearrowleft_{\sigma_K} \mathbb{R}$

Arithmetics of K is reflected to the dynamics of (A_K, σ_K)

1. The partition function $= \zeta_K$.
2. $G_K^{\mathrm{ab}} \curvearrowright (A_K, \sigma_K)$.
3. For $\beta \leq 1$, $|\mathrm{KMS}_\beta(A_K, \sigma_K)| = 1$.
4. For $\beta > 1$, $G_K^{\mathrm{ab}} \curvearrowright \mathrm{exKMS}_\beta(A_K, \sigma_K)$ is free and transitive.
5. There is a K-subalgebra $A_0 \subset A$ such that $\varphi(A_0) = K^{\mathrm{ab}}$ for $\varphi \in \mathrm{KMS}_\beta(A_K, \sigma_K)$.

(Bost–Connes '95, Connes–Marcolli–Ramachandran'05, Ha–Paugam'05, Laca–Larsen–Neshveyev'09, Yalkinoglu'13)

Classification of (A_K, σ_K): motivation

Neukirch–Uchida theorem

$$\text{If } G_K \cong G_L, \text{ then } K \cong L.$$

However, $G_K^{\mathrm{ab}} \cong G_L^{\mathrm{ab}}$ does not imply $K \cong L$.

Question(Cornelissen–Marcolli '11)

$$(A_K, \sigma_K) \cong (A_L, \sigma_L) \Rightarrow K \cong L?$$

- $(A_K, \sigma_K) \cong (A_L, \sigma_L) \Rightarrow \zeta_K = \zeta_L$.
  (If both K and L are Galois extensions, then $\zeta_K = \zeta_L \Rightarrow K = L$).
- (Takeishi'16) If $A_K \cong A_L$, then $\zeta_K = \zeta_L$ and $h_K^1 = h_L^1$.

Another question

$$A_K \cong A_L \Leftrightarrow \zeta_K \cong \zeta_L \text{ and } h_K^1 = h_L^1?$$

At first we expected...

$$
\begin{tikzcd}[column sep=large, row sep=large]
\begin{matrix}\text{number field}\\ K\end{matrix} \arrow[r] & \begin{matrix}\text{semigroup action}\\ Y_K \curvearrowleft I_K\end{matrix} \arrow[r] \arrow[l, bend right, "\text{number theory?}"'] \arrow[d] & \begin{matrix}\text{groupoid}\\ Y_K \rtimes I_K\end{matrix} \arrow[d] & \\
 & \begin{matrix}\text{C*-dynamics}\\ (A_K, \sigma_K)\end{matrix} \arrow[r] \arrow[ur, "\text{Renault's theorem?}"] & \begin{matrix}\text{C*-algebra}\\ A_K\end{matrix} \arrow[r] \arrow[l, bend left, "\text{classification of group actions?}"] & \text{K-theory} \arrow[l, bend left, "\text{classification of nonsimple C*-algebras?}"]
\end{tikzcd}
$$

- $A_K = C(Y_K) \rtimes I_K \curvearrowleft \mathbb{T}^{\mathcal{P}_K}$: dual action. σ_K is its restriction to a subgroup $\mathbb{R} \subset \mathbb{T}^{\mathcal{P}_K}$.
- Renault's theorem: if $C^*(\mathcal{G}_K) \cong C^*(\mathcal{G}_L)$ induces $C(\mathcal{G}_K^0) \cong C(\mathcal{G}_L^0)$, then $\mathcal{G}_K \cong \mathcal{G}_L$.

What we actually did

$$
\begin{tikzcd}
\begin{array}{c}\text{number field}\\ K\end{array} \arrow[r] & \begin{array}{c}\text{semigroup action}\\ Y_K \curvearrowleft I_K\end{array} \arrow[l, bend right, "\substack{\text{Cornelissen–de Smit–}\\ \text{Li–Marcolli–Smit '17}}"'] \arrow[r] \arrow[d] & \begin{array}{c}\text{groupoid}\\ Y_K \rtimes I_K\end{array} \arrow[d] & \\
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Main theorem

Theorem (K.-Takeishi '17)

The following are equivalent:

1. $K \cong L$ *as fields,*
2. $Y_K \curvearrowleft I_K \cong Y_L \curvearrowleft I_L$ *as semigroup actions,*
3. $Y_K \rtimes I_K \cong Y_L \rtimes I_L$ *as groupoids,*
4. $(A_K, \sigma_K) \cong (A_L, \sigma_L)$ *as* $\mathbb{R}$*-C*-algebras,*
5. $A_K \cong A_L$ *as C*-algebras,*
6. A_K *and* A_L *are* $\mathrm{KK}(X)$*-equivalent* $+\alpha$*,*
7. A_K *and* A_L *have the same "K-theoretic data".*

What we actually show is (7) $\Rightarrow$ (2).

Toeplitz algebra and generalizations

- The Toeplitz algebra $\mathcal{T} = C^*(s) \subset \mathbb{B}(\ell^2\mathbb{N})$ is written as the semigroup crossed product $C(\mathbb{N}^+) \rtimes \mathbb{N}$.

$$\begin{tikzcd} \mathbb{C} \arrow[r, bend left] & \mathbb{C} \arrow[r, bend left] & \mathbb{C} \arrow[r, bend left] & \mathbb{C} \arrow[r, bend left] & \cdots & \mathbb{C} \arrow[loop above] \end{tikzcd}$$

- For $A \curvearrowleft \mathbb{Z}$, let $\mathcal{T}(A, \alpha) = (C(\mathbb{N}^+) \otimes A) \rtimes \mathbb{N}$.

$$\begin{tikzcd} A \arrow[r, bend left] & A \arrow[r, bend left] & A \arrow[r, bend left] & A \arrow[r, bend left] & \cdots & A \arrow[loop above] \end{tikzcd}$$

  Then $0 \to A \otimes \mathbb{K} \to \mathcal{T}(A, \alpha) \to A \rtimes \mathbb{Z} \to 0$ is exact.

- If we have $A \to B$, there is a semigroup crossed product $\mathcal{T}(A, \alpha, B)$

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Multi-Toeplitz dynamical system

If we have a compatible family of C*-algebras $A_F \curvearrowleft \mathbb{Z}^{\mathcal{P}\setminus F}$ for $F \subset \mathcal{P}$ and $\mathbb{Z}^{\mathcal{P}\setminus F'}$-$*$-homomorphisms $A_F \to A_{F'}$ for $F \subset F'$, then we have a $\mathbb{N}^{\mathcal{P}}$-action on a continuous field of C*-algebras $\mathcal{A} = \bigsqcup A_F$.

$$
\begin{tikzcd}
A_{\{1,2\}} \arrow[r] \arrow[d] & A_{\{1,2\}} \arrow[r] \arrow[d] & A_{\{1,2\}} \arrow[r] \arrow[d] & \cdots & A_{\{1\}} \arrow[loop right] \arrow[d] \\
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\vdots \arrow[d] & \vdots \arrow[d] & \vdots & \ddots & \vdots \\
A_{\{2\}} \arrow[r] \arrow[loop below] & A_{\{2\}} \arrow[r] \arrow[loop below] & A_{\{2\}} \arrow[r] \arrow[loop below] & \cdots & A_{\emptyset} \arrow[loop right] \arrow[loop below]
\end{tikzcd}
$$

In particular we are interested in the case that $|\mathcal{P}| = \infty$.

Ideal structure and irreducible subquotients

Let $\mathcal{A} \curvearrowleft \mathbb{N}^{\mathcal{P}}$ be a multi-Toeplitz dynamical system and let $\mathfrak{A} := \mathcal{A} \rtimes \mathbb{N}^{\mathcal{P}}$.

- A $\mathbb{N}^{\mathcal{P}}$-invariant open subset U of $(\mathbb{N}^{+})^{\mathcal{P}}$ corresponds to an ideal $\mathfrak{A}(U)$.
- ($\mathbb{N}^{\mathcal{P}}$-invariant open subsets of $(\mathbb{N}^{+})^{\mathcal{P}}$) $\leftrightarrow$ (open subsets of $2^{\mathcal{P}}$), where 2 is equipped with the topology $\{\emptyset, \{0\}, 2\}$.
- An locally closed subset (i.e. a subset of the form $Z = U \setminus V$) of $2^{\mathcal{P}}$ corresponds to a subquotient $\mathfrak{A}(Z)$.
- When $\mathcal{P}$ is finite, $\{F\}$ is locally closed and $\mathfrak{A}(\{F\}) \cong A_F \rtimes \mathbb{Z}^{\mathcal{P} \setminus F} \otimes \mathbb{K}$. When $\mathcal{P}$ is infinite, $\{F\}$ is locally closed iff F is finite.
- By continuity, we have $A_S = \varinjlim_{F \subset\subset S} A_F$. Hence a multi-Toeplitz dynamical system is reconstructed from the family $\{A_F\}_{F \subset\subset \mathcal{P}}$.
- If we want to reconstruct $\mathcal{A} \curvearrowleft \mathbb{N}^{\mathcal{P}}$ from $\mathfrak{A}$, the remaining task is to reconstruct $A_F \curvearrowleft \mathbb{Z}^{\mathcal{P} \setminus F}$ from $\mathfrak{A}(\{F\}) \sim A_F \rtimes \mathbb{Z}^{\mathcal{P} \setminus F}$.

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Algebraic number theory

- K: number field i.e. $[K:\mathbb{Q}] < \infty$
- $\mathcal{O}_K$: the ring of integers
- $\mathcal{P}_K$: set of prime ideals of $\mathcal{O}_K$
- $I_K = \{\text{ideals of } \mathcal{O}_K\} \cong \bigoplus_{\mathcal{P}_K} \mathfrak{p}^{\mathbb{N}}$ (prime ideal factorization).
- $J_K := \bigoplus \mathfrak{p}^{\mathbb{Z}}$
- $K_{\mathfrak{p}}$: local field (e.g. $\mathbb{Q}_p$), $\mathcal{O}_{\mathfrak{p}}$: ring of integers (e.g. $\mathbb{Z}_p$)
- valuation: $v_{\mathfrak{p}} \colon K_{\mathfrak{p}} \to \mathbb{Z} \cup \{+\infty\}$, $v_{\mathfrak{p}}^{-1}(1) = \mathcal{O}_{\mathfrak{p}}^*$, $K_{\mathfrak{p}}^* \cong \mathcal{O}_{\mathfrak{p}}^* \times \mathbb{Z}$.
- $\hat{\mathcal{O}}_K^* := \prod \mathcal{O}_{\mathfrak{p}}^*$ and $\mathbb{A}_{K,f}^* := \prod'(K_{\mathfrak{p}}^*, \mathcal{O}_{\mathfrak{p}}^*)$. That is,

$$1 \to \hat{\mathcal{O}}_K^* \to \mathbb{A}_{K,f}^* \to J_K \to 1$$

- Class field theory: The Artin reciprocity map $\phi \colon \mathbb{A}_{K,f}^* \to G_K^{\mathrm{ab}}$ induces an isomorphism $\mathbb{A}_{K,f}^* / \overline{K^*} \cong G_K^{\mathrm{ab}}$.

The space Y_K

- $0 \to \hat{\mathcal{O}}_K^* \to \mathbb{A}_{K,f}^* \to J_K \to 1$ restricts to

$$0 \to \hat{\mathcal{O}}_K^* \to \hat{\mathcal{O}}_K^\natural \to I_K \to 0$$

- Consider actions $\hat{\mathcal{O}}_K^\natural \curvearrowright \hat{\mathcal{O}}_K$ and $\phi \colon \hat{\mathcal{O}}_K^\natural \to G_K^{\mathrm{ab}}$
- Set $Y_K := (\hat{\mathcal{O}}_K \times G_K^{\mathrm{ab}})/\hat{\mathcal{O}}_K^* \curvearrowleft I_K$.

Consider the semigroup action $C(Y_K) \curvearrowleft I_K$.

- $\prod v_\mathfrak{p} \colon \hat{\mathcal{O}}_K \to (\mathbb{N}^+)^{\mathcal{P}_K}$ induces the map $\mathrm{val} \colon Y_K \to (\mathbb{N}^+)^{\mathcal{P}_K}$
- Then $C(Y_K) \curvearrowleft I_K$ is a multi-Toeplitz dynamical system with the fibers $C(G_K^F)$, where $G_K^F := G_K^{\mathrm{ab}}/\phi(\prod_{\mathfrak{p} \in F} \mathcal{O}_\mathfrak{p}^*)$.

Let $A_K := C(Y_K) \rtimes I_K$.

- For finite $F \subset \mathcal{P}_K$, we have a subquotient $A_K(\{F\})$.
- Let $J_K^F := \bigoplus_{\mathfrak{p} \notin F} \mathbb{Z}^\mathfrak{p}$. Then $A_K(\{F\}) \cong C(G_K^F) \rtimes J_K^F \otimes \mathbb{K}$.
- According to class fiend theory, G_K^F is a pro-$\mathcal{N}_F$ group for some finite subset $\mathcal{N}_F \subset \mathcal{P}_\mathbb{Q}$.

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Consider the semigroup action $C(Y_K) \curvearrowleft I_K$.

- $\prod v_\mathfrak{p} \colon \hat{\mathcal{O}}_K \to (\mathbb{N}^+)^{\mathcal{P}_K}$ induces the map $\mathrm{val} \colon Y_K \to (\mathbb{N}^+)^{\mathcal{P}_K}$
- Then $C(Y_K) \curvearrowleft I_K$ is a multi-Toeplitz dynamical system with the fibers $C(G_K^F)$, where $G_K^F := G_K^{\mathrm{ab}}/\phi(\prod_{\mathfrak{p} \in F} \mathcal{O}_\mathfrak{p}^*)$.

Let $A_K := C(Y_K) \rtimes I_K$.

- For finite $F \subset \mathcal{P}_K$, we have a subquotient $A_K(\{F\})$.
- Let $J_K^F := \bigoplus_{\mathfrak{p} \notin F} \mathbb{Z}^\mathfrak{p}$. Then $A_K(\{F\}) \cong C(G_K^F) \rtimes J_K^F \otimes \mathbb{K}$.
- According to class fiend theory, G_K^F is a pro-$\mathcal{N}_F$ group for some finite subset $\mathcal{N}_F \subset \mathcal{P}_\mathbb{Q}$.

The "K-theoretic data"

In summary, the Bost–Connes C*-algebra A_K is the semigroup crossed product of a multi-Toeplitz dynamical system. Hence we get a subquotient

$$B_K^F := A_K(\{F\}) \sim C(G_K^F) \rtimes J_K^F$$

for each finite subset $F \subset \mathcal{P}_K$.
Let $F_{\mathfrak{p}} = F \cup \{\mathfrak{p}\}$. Then

$$0 \to B_K^{F_{\mathfrak{p}}} \to A_K(\{F, F_{\mathfrak{p}}\}) \to B_K^F \to 0.$$

Definition

The K-theoretic data of A_K is $\{\mathrm{K}_*(B_K^F)\}$ and $\{\partial\colon \mathrm{K}_*(B_K^F) \to \mathrm{K}_{*+1}(B_K^{F_{\mathfrak{p}}})\}$.

Proposition

If $\varphi\colon A_K \cong A_L$, then there is $\chi\colon \mathcal{P}_K \cong \mathcal{P}_L$ such that φ induces $B_K^F \cong B_L^{\chi(F)}$.

Reconstruction of profinite completions

Theorem (K.–Takeishi'17)

Let Γ be a free abelian group and let $\varphi \colon \Gamma \to G$ be its pro-$\mathcal{N}$ completion. Then, φ is reconstructed from the inclusion $\mathrm{K}_*(C^*_r\Gamma) \to \mathrm{K}_*(C(G) \rtimes \Gamma)$.

Corollary

Let Γ_1 and Γ_2 be free abelian groups and let $\varphi_i \colon \Gamma_i \to G_i$ are pro-$\mathcal{N}$ completions. If we have $f : \Gamma_1 \cong \Gamma_2$ and

$$\begin{tikzcd}
\mathrm{K}_*(C^*\Gamma_1) \arrow[r] \arrow[d] & \mathrm{K}_*(C^*\Gamma_2) \arrow[d] \\
\mathrm{K}_*(C(G) \rtimes \Gamma) \arrow[r] & \mathrm{K}_*(C(G) \rtimes \Gamma)
\end{tikzcd}$$

then there is an isomorphism $F \colon G_1 \to G_2$ such that $F \circ \varphi_1 = \varphi_2 \circ f$.

Example 1: $C(\mathbb{Z}_p) \rtimes \mathbb{Z}$

Let $\Gamma = \mathbb{Z}$, $G = \mathbb{Z}_p$ and $\varphi(1) = 1$. Then, $C(G) = \varinjlim C(G_n)$, where $G_n = \mathbb{Z}/p^n\mathbb{Z}$.

$$\mathrm{K}_0(C(G_n) \rtimes \Gamma) \cong \mathbb{Z},$$
$$\mathrm{K}_1(C(G_n) \rtimes \Gamma) \cong \mathbb{Z}.$$

The map $G_{n+1} \to G_n$ induces

$$p \colon \mathrm{K}_0(C(G_n) \rtimes \Gamma) \to \mathrm{K}_0(C(G_{n+1}) \rtimes \Gamma)$$
$$1 \colon \mathrm{K}_1(C(G_n) \rtimes \Gamma) \to \mathrm{K}_1(C(G_{n+1}) \rtimes \Gamma)$$

Therefore, $\mathrm{K}_0(C(\mathbb{Z}_p) \rtimes \mathbb{Z}) \cong \mathbb{Z}[p^{-1}]$ and $\mathrm{K}_1(C(\mathbb{Z}_p) \rtimes \mathbb{Z}) \cong \mathbb{Z}$.

Example 2: $C(\mathbb{Z}_p) \rtimes \mathbb{Z}^2$

Let $\Gamma = \mathbb{Z}^2$, $G = \mathbb{Z}_p$ and $\varphi(1,0) = 1$ and $\varphi(0,1) = 1 + p + p^2 + \dots$.
Then, $C(G) = \varinjlim C(G_n)$, where $G_n = \mathbb{Z}/p^n\mathbb{Z}$.

$$\mathrm{K}_0(C(G_n) \rtimes \Gamma) \cong \mathbb{Z} \oplus \mathbb{Z},$$
$$\mathrm{K}_1(C(G_n) \rtimes \Gamma) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

The map $G_{n+1} \to G_n$ induces

$$\mathrm{diag}(p,1) \colon \mathrm{K}_0(C(G_n) \rtimes \Gamma) \to \mathrm{K}_0(C(G_{n+1}) \rtimes \Gamma)$$
$$\begin{pmatrix} p & -1 \\ 0 & 1 \end{pmatrix} \colon \mathrm{K}_1(C(G_n) \rtimes \Gamma) \to \mathrm{K}_1(C(G_{n+1}) \rtimes \Gamma)$$

Therefore,

$$\mathrm{K}_0(C(\mathbb{Z}_p) \rtimes \mathbb{Z}^2) \cong \mathbb{Z}[p^{-1}] \oplus \mathbb{Z},$$
$$\mathrm{K}_1(C(\mathbb{Z}_p) \rtimes \mathbb{Z}^2) \cong \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathbb{Z} + \bigcup \begin{pmatrix} \sum_{j=1}^{n-1} p^{-j} \\ -p^{-n} \end{pmatrix} \mathbb{Z} \subset \mathbb{Z}[p^{-1}]^2.$$

Here, $\ker(\Gamma \to G_n) = \mathrm{K}_1(C^*\mathbb{Z}^2) \cap p^n \mathrm{K}_1(C(\mathbb{Z}_p) \rtimes \mathbb{Z}^2)$.

Sketch of the proof

- The group $\mathrm{K}_*(C^*\Gamma)$ is isomorphic to $\bigwedge^*\Gamma$.
- For a finitely generated subgroup $\Pi = \mathbb{Z}v_1 \oplus \cdots \oplus \mathbb{Z}v_k$, let $\beta_\Pi := \beta_{v_1} \wedge \cdots \wedge \beta_{v_k}$.
- Then,

$$\delta(\beta_\Pi) := \max\{n \in \mathbb{Z}_{>0} \mid n^{-1}\beta_\Pi \in \mathrm{K}_*(C(G) \rtimes \Gamma)\}$$

  coincides with $|G/\overline{\varphi(\Pi)}|$.
- By using the data $\delta(\beta_\Pi)$, we can reconstruct the subgroup $\ker(\Gamma \to G_n) \subset \Gamma$ as an inclusion of subgroups of $\mathrm{K}_*(C(G) \rtimes \Gamma)$.
- General pro-$\mathcal{N}$ group G is isomorphic to $G_{p_1} \times \cdots \times G_{p_k}$, where G_{p_i} is a pro-p_i group and $\mathcal{N} = \{p_1, \dots, p_k\}$.
- If we have isomorphisms $\mathrm{K}_*(B_K^F) \cong \mathrm{K}_*(B_L^{\chi(F)})$ compatible with ∂, then is must be compatible with $\chi\colon \mathrm{K}_*(C^*J_K) \cong \mathrm{K}_*(C^*J_L)$.