Connes Integration Formula for Noncommutative Euclidean Space

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M_f, ∇ and Δ

Let ∇ denotes the gradient on $L_2(\mathbb{R}^d)$, that is ∇ is the d-tuple $-i(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_d})$ of operators $\frac{\partial}{\partial x_j}$ of partial differentiation on $L_2(\mathbb{R}^d)$. Let M_f denotes the multiplication operator on $L_2(\mathbb{R}^d)$ by a function f. The Laplace operator $-\Delta$ is formally defined as ∇^2. Connes trace formula (or Connes integration formula) involves operator $(1-\Delta)^{-d/2}$ which can be also understood as the operator $g(\nabla)$ defined with the help of usual functional calculus, where $g(t) = (1+|t|^2)^{-d/2}$, $t \in \mathbb{R}^d$.

One of the most beautiful results in Noncommutative Geometry concerns so-called Cwikel estimates for the singular values of the operator $M_f g(\nabla)$ on $L_2(\mathbb{R}^d)$ in weak Schatten ideals $\mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d))$.

Connes Integration Formula: the beginning

For any smooth compactly supported function f on $\mathbb{R}^d$, Connes established the following formula

$$\mathrm{tr}_\omega(M_f(1-\Delta)^{-\frac{d}{2}}) = c_d \int_{\mathbb{R}^d} f(s)ds.$$

Here, tr_ω is an arbitrary Dixmier trace (to be defined later).
This formula serves as a motivation for the general notion of the noncommutative integral and is ubiquitous in noncommutative geometry.
In this talk, we provide a substantially stronger (and more general) version of the Connes Integration Formula for noncommutative Euclidean space.

General notations

Fix throughout a separable infinite dimensional Hilbert space H. We let $B(H)$ denote the algebra of all bounded operators on H. For a compact operator T on H, let $\mu(k, T)$ denote $k-$th largest singular value (these are the eigenvalues of $|T|$). The sequence $\mu(T) = \{\mu(k, T)\}_{k \geq 0}$ is referred to as to the singular value sequence of the operator T. The standard trace on $B(H)$ is denoted by tr.

Fix an orthonormal basis in H (the particular choice of a basis is inessential). We identify the algebra ℓ_∞ of bounded sequences with the subalgebra of all diagonal operators with respect to the chosen basis. For a given sequence $\alpha \in \ell_\infty$, we denote the corresponding diagonal operator by $\mathrm{diag}(\alpha)$.

Principal ideals $\mathcal{L}_{p,\infty}$

Let $\mathcal{L}_{p,\infty}$ be the principal ideal in $B(\ell_2)$ generated by the element $A_0 = \mathrm{diag}(\{(k+1)^{-\frac{1}{p}}\}_{k\geq 0})$. Equivalently,

$$\mathcal{L}_{p,\infty} = \{A: \sup_{k\geq 0}(k+1)^{\frac{1}{p}}\mu(k,A) < \infty\}.$$

In Noncommutative Geometry, a compact operator A is called an infinitesimal of order $\frac{1}{p}$ if

$$\mu(k,A) = O((k+1)^{-\frac{1}{p}}), \quad k \in \mathbb{Z}_+.$$

In other words, $\mathcal{L}_{p,\infty}$ is the set of all infinitesimals of order $\frac{1}{p}$.

Traces on $\mathcal{L}_{1,\infty}$

Definition

A linear functional $\varphi : \mathcal{L}_{1,\infty} \to \mathbb{C}$ is called a trace if $\varphi(AB) = \varphi(BA)$ for every $A \in \mathcal{L}_{1,\infty}$ and for every $B \in B(H)$.

The trace φ is called normalised if $\varphi(\mathrm{diag}(\{\frac{1}{k+1}\}_{k\geq 0})) = 1$.
There exists a plethora of (normalised) traces on $\mathcal{L}_{1,\infty}$. The most famous ones are Dixmier traces.

Dixmier traces

An extended limit is a functional ω in ℓ_∞ which extends the "limit" functional on the subspace c of convergent sequences and $\|\omega\|_{\ell_\infty^*} = 1$. Extended limits exist by the Hahn-Banach theorem.

Definition (Dixmier)

If ω is an extended limit then the functional

$$A \to \omega\left(\frac{1}{\log(n+2)}\sum_{k=0}^{n}\mu(k,A)\right), \quad 0 \leq A \in \mathcal{L}_{1,\infty}$$

is finite and additive on the positive cone of $\mathcal{L}_{1,\infty}$. Thus, it uniquely extends to a unitarily invariant linear functional on $\mathcal{L}_{1,\infty}$. The latter is called a Dixmier trace and is denoted by tr_ω.

Basic properties of traces

1. Every Dixmier trace is positive
2. Every positive trace is continuous with respect to the natural quasi-norm on $\mathcal{L}_{1,\infty}$
3. Every continuous trace is a linear combination of 4 positive traces.
4. There are positive traces which are not Dixmier traces
5. There exist discontinuous traces
6. There are $2^{2^{\mathbb{N}}}$ positive traces

More information on the traces is available in [LSZ].

Cwikel estimates in $\mathcal{L}_{1,\infty}$

We say that $f \in \ell_1(L_2)(\mathbb{R}^d)$ if

$$\sum_{k \in \mathbb{Z}^d} \|f\chi_{k+K}\|_2 < \infty.$$

Here, $K = [0,1]^d$.
The result below is (probably) due to Birman and Solomyak and is so far the best available. However, for the proof of this result we refer to [LeSZ].

Theorem

If $f \in \ell_1(L_2)(\mathbb{R}^d)$, then

$$M_f(1-\Delta)^{-\frac{d}{2}} \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d)).$$

Connes Integration Formula II

The following result established in [CZ-JOT] (Section 4 there is fully devoted to the proof of this result).

Theorem

If $f \in \ell_1(L_2)(\mathbb{R}^d)$, then

$$\varphi(M_f(1-\Delta)^{-\frac{d}{2}}) = c_d \int_{\mathbb{R}^d} f(s)ds$$

for every continuous normalised trace on $\mathcal{L}_{1,\infty}$.

Observe that we have significantly weakened the assumption imposed on the the function f comparatively with the original version due to Connes and significantly extended to stock of singular traces for which the equality above holds.

What is Noncommutative Euclidean space I

In Noncommutative Geometry, we replace the space with the algebra of functions on this space. So, $\mathbb{R}^d$ is the $*-$algebra generated by elements $\{x_k\}_{k=1}^d$ satisfying the conditions

$$x_k = x_k^*, \quad 1 \leq k \leq d,$$

$$[x_{k_1}, x_{k_2}] = 0, \quad 1 \leq k_1, k_2 \leq d.$$

Now, let us distort the latter relations as follows:

$$[x_{k_1}, x_{k_2}] = i\theta_{k_1,k_2}, \quad 1 \leq k_1, k_2 \leq d.$$

Here, $\theta \in M_d(\mathbb{R})$ is some skew-symmetric matrix.
It is natural to treat the algebra generated by the latter $\{x_k\}_{k=1}^d$ as the (algebra of functions on the) Noncommutative Euclidean space.

What is Noncommutative Euclidean space II

In $\mathbb{R}^d$, the (coordinate) functions $\{x_k\}_{k=1}^d$ are unbounded. So are the elements $\{x_k\}_{k=1}^d$ on the Noncommutative Euclidean space. It is inconvenient to operate with unbounded operators (issues with domains and the like). The standard procedure is to pass to the respective unitary groups.
For $t \in \mathbb{R}^d$, set

$$U(t) = \exp(i\sum_{k=1}^d t_k x_k).$$

A formal manipulation with $U(t)'$s (explained on the next slide) yields

$$U(t+s) = e^{-\frac{i}{2}\langle t,\theta s\rangle} U(t)U(s), \quad t,s \in \mathbb{R}^d.$$

Motivation for the definition of noncommutative Euclidean space

Baker-Campbell Hausdorff formula states that (at the level of formal power series)

$$\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y] + \cdots)$$

where the "$\cdots$" above means terms involving commutators with $[X, Y]$. Formally plugging in $X = it_j x_j$ and $Y = it_k x_k$ for $t_j, t_k \in \mathbb{R}$, we then get

$$\begin{aligned}\exp(it_j x_j)\exp(it_k x_k) &= \exp(it_j x_j + it_k x_k + \frac{1}{2}[it_j x_j, it_k x_k] + \cdots) \\ &= \exp(it_j x_j + it_k x_k + \frac{i}{2}\theta_{j,k} t_j t_k).\end{aligned}$$

More generally, if we denote for a vector $t = (t_1, \dots, t_d) \in \mathbb{R}^d$, $U(t) := \exp(it_1 x_1 + it_2 x_2 + \cdots + it_d x_d)$, then we have the formal relation,

$$U(t + s) = \exp(\frac{i}{2}\langle t, \theta s\rangle)U(t)U(s).$$

Algebra of Canonical Commutation Relations

The following definition is from Bratteli and Robinson.

Definition

Let H be a real Hilbert space and let σ be a symplectic bilinear form (that is a non-degenerate bilinear mapping on $H \times H$ satisfying $\sigma(f, g) + \sigma(g, f) = 0$ for all $f, g \in H$). CCR algebra is the universal C^*-algebra generated by the unitary operators $\{U(f)\}_{f \in H}$ satisfying the conditions

$$U(f+g) = e^{-\frac{i}{2}\sigma(f,g)} U(f)U(g), \quad f, g \in H.$$

When $H = \mathbb{R}^d$ and

$$\sigma(t, s) = \langle t, \theta s \rangle, \quad t, s \in \mathbb{R}^d,$$

we obtain the definition of a Noncommutative Euclidean space $\mathbb{R}^d_\theta$.

Formal definition of Noncommutative Euclidean space

We use a concrete representation of the Noncommutative Euclidean Space. Namely, we set

$$(U(t)\xi)(u) = e^{-\frac{i}{2}\langle t,\theta u\rangle}\xi(u-t), \quad \xi \in L_2(\mathbb{R}^d), \quad u,t \in \mathbb{R}^d. \tag{1}$$

Definition

The von Neumann subalgebra in $B(L_2(\mathbb{R}^d))$ generated by $\{U(t)\}_{t\in\mathbb{R}^d}$, introduced in (1), is called the Noncommutative Euclidean space and denoted by $L_\infty(\mathbb{R}^d_\theta)$.

The map $t \to U(t)$, $t \in \mathbb{R}^d$, is the twisted left regular representation of the Abelian group $\mathbb{R}^d$ for the 2-cocycle $\omega : (t,u) \to \exp(\frac{i}{2}\langle t, \theta s\rangle)$. The von Neumann algebra $L_\infty(\mathbb{R}^d_\theta)$ is exactly the twisted group von Neumann algebra $W^*(\mathbb{R}^d, \omega)$ for the group 2-cocycle ω.

Faithful normal semifinite trace on $L_\infty(\mathbb{R}^d_\theta)$

The following assertion is well-known. In [LeSZ], a *spatial* isomorphism is constructed.

Theorem

For every non-degenerate antisymmetric real matrix θ, the algebra $L_\infty(\mathbb{R}^d_\theta)$ is isomorphic to $B(L_2(\mathbb{R}^{\frac{d}{2}}))$.

Having established the isomorphism between $r : L_\infty(\mathbb{R}^d_\theta) \to B(L_2(\mathbb{R}^{\frac{d}{2}}))$ we now equip $L_\infty(\mathbb{R}^d_\theta)$ with a faithful normal semifinite trace $\tau_\theta = \mathrm{Tr} \circ r$. We can now define L_p-spaces on $L_\infty(\mathbb{R}^d_\theta)$.

$$L_p(\mathbb{R}^d_\theta) = \Big\{ x \in L_\infty(\mathbb{R}^d_\theta) : \ \tau_\theta(|x|^p) < \infty \Big\}.$$

Partial derivatives on $L_\infty(\mathbb{R}^d_\theta)$ I

In a sense, our picture extends the Fourier dual of the classical definition of the Moyal plane. So what-should-be differentiation operator in the classical setting is multiplication operator for us.
Let D_k, $1 \le k \le d$ be multiplication operators on $L_2(\mathbb{R}^d)$

$$(D_k\xi)(t) = t_k\xi(t), \quad \xi \in L_2(\mathbb{R}^d).$$

For brevity, we denote $\nabla = (D_1, \cdots, D_d)$ and $-\Delta = \sum_{k=1}^d D_k^2$. For every $1 \le k \le d$, we have

$$[D_k, U(s)] = s_k U(s), \quad s \in \mathbb{R}^d. \tag{2}$$

Partial derivatives on $L_\infty(\mathbb{R}^d_\theta)$ II

If $[D_k, x] \in B(L_2(\mathbb{R}^d))$ for some $x \in L_\infty(\mathbb{R}^d_\theta)$, then $[D_k, x] \in L_\infty(\mathbb{R}^d_\theta)$. This crucial fact allows us to introduce mixed partial derivative $\partial^\alpha x$ of $x \in L_\infty(\mathbb{R}^d_\theta)$.

Definition

Let α be a multiindex and let $x \in L_\infty(\mathbb{R}^d_\theta)$. If every repeated commutator $[D_{\alpha_j}, [D_{\alpha_j+1}, \cdots, [D_{\alpha_n}, x]]]$, $1 \le j \le n$, is a bounded operator on $L_2(\mathbb{R}^d)$, then the mixed partial derivative $\partial^\alpha x$ of x is defined as

$$\partial^\alpha x = [D_{\alpha_1}, [D_{\alpha_2}, \cdots, [D_{\alpha_n}, x]]].$$

In this case, we have that $\partial^\alpha x \in L_\infty(\mathbb{R}^d_\theta)$. As usual, $\partial^0 x = x$.

Sobolev spaces in $L_\infty(\mathbb{R}^d_\theta)$

We can introduce the Sobolev space $W^{m,p}(\mathbb{R}^d_\theta)$ associated with the noncommutative plane in the following way.

Definition

For $m \in \mathbb{Z}_+$ and $p \geq 1$, the space $W^{m,p}(\mathbb{R}^d_\theta)$ is the space of $x \in L_p(\mathbb{R}^d_\theta)$ such that every partial derivative of x up to order m is also in $L_p(\mathbb{R}^d_\theta)$. This space is equipped with the norm,

$$\|x\|_{W^{m,p}} = \sum_{|\alpha| \leq m} \|\partial^\alpha x\|_p, \quad x \in W^{m,p}(\mathbb{R}^d_\theta).$$

Cwikel estimates in the Noncommutative Euclidean space

The following Cwikel-type estimate for the ideal $\mathcal{L}_{1,\infty}$ and for the Noncommutative Euclidean space is established in [LeSZ].

Theorem

If $x \in W^{d,1}(\mathbb{R}^d_\theta)$, then

$$x(1-\Delta)^{-\frac{d}{2}} \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d)).$$

Connes Integration Formula in the Noncommutative Euclidean space

Theorem below is the main result of this talk. It is proved in [SZ-cmp].

Theorem

If $x \in W^{d,1}(\mathbb{R}^d_\theta)$, then

$$\varphi(x(1-\Delta)^{-\frac{d}{2}}) = \tau_\theta(x)$$

for every normalised trace φ on $\mathcal{L}_{1,\infty}$.

Idea of the proof I

Since φ is unitarily invariant, it follows that

$$\varphi(x(1-\Delta)^{-\frac{d}{2}}) = \varphi(e^{i\langle t,\nabla\rangle}x(1-\Delta)^{-\frac{d}{2}}e^{-i\langle t,\nabla\rangle}), \quad t\in\mathbb{R}^d.$$

Since $e^{-i\langle t,\nabla\rangle}$ commutes with $(1-\Delta)^{-\frac{d}{2}}$, it follows that

$$\varphi(x(1-\Delta)^{-\frac{d}{2}}) = \varphi(e^{i\langle t,\nabla\rangle}xe^{-i\langle t,\nabla\rangle}(1-\Delta)^{-\frac{d}{2}}).$$

For every $x\in L_\infty(\mathbb{R}^d_\theta)$, we have

$$e^{i\langle t,\nabla\rangle}xe^{-i\langle t,\nabla\rangle} = U(-\theta^{-1}t)xU(\theta^{-1}t).$$

Therefore,

$$\varphi(x(1-\Delta)^{-\frac{d}{2}}) = \varphi(U(-\theta^{-1}t)xU(\theta^{-1}t)(1-\Delta)^{-\frac{d}{2}}).$$

Idea of the proof II

Consider the functional

$$F : x \to \varphi(x(1-\Delta)^{-\frac{d}{2}}), \quad x \in W^{d,1}(\mathbb{R}^d_\theta).$$

It is a well defined bounded linear functional on $W^{d,1}(\mathbb{R}^d_\theta)$. We have

$$F(x) = F(U(-\theta^{-1}t)xU(\theta^{-1}t)), \quad x \in L_\infty(\mathbb{R}^d_\theta), \quad t \in \mathbb{R}^d.$$

1. We show that F is bounded in the norm of $L_1(\mathbb{R}^d_\theta)$ (not just $W^{d,1}(\mathbb{R}^d_\theta)$).
2. We show that every bounded functional on $L_1(\mathbb{R}^d_\theta)$ satisfying the latter condition is proportional to τ_θ.

1. [NCG] Connes A. *Noncommutative Geometry.*
2. [LeSZ] Levitina G., Sukochev F., Zanin D. *Cwikel estimates revisited.*
3. [LSZ] Lord S., Sukochev F., Zanin D. *Singular traces. Theory and Applications.* De Gruyter Studies in Mathematics, vol. **46**, De Gruyter, Berlin, (2013)
4. [CZ-JOT] Sukochev F., Zanin D. *A C^*-algebraic approach to the principal symbol I.* , JOT, to appear.
5. [SZ-cmp] Sukochev F., Zanin D. *Connes Integration Formula for the noncommutative plane.* Commun. Math. Phys. **359**, 449–466 (2018)

Thank you for your attention