Maslov-Type Index Theory For Symplectic Paths With Lagrangian Boundary Conditions

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Linear symplectic spaces–1

Let V be an $2m$-dimensional vector space over $\mathbf{R}$ with its dual space V^*, and let $\tilde{\omega} : V \times V \to \mathbf{R}$ be a bilinear form.

Definition *The map $\tilde{\omega}^* : V \to V^*$ is the linear map defined by $\tilde{\omega}^*(v)(u) = \tilde{\omega}(v, u)$.*

The kernel of $\tilde{\omega}^*$ is the subspace U of V

Definition *A skew-symmetric bilinear map $\tilde{\omega}$ is symplectic if $\tilde{\omega}^*$ is bijective, i.e., $U = \{0\}$. The map $\tilde{\omega}$ is then called a linear symplectic structure (or symplectic form) on V, and $(V, \tilde{\omega})$ is called a symplectic vector space.*

We define

$$E^{\perp_{\tilde{\omega}}} = \{u \in V \,|\, \tilde{\omega}(u, v) = 0, \ \ \forall\, v \in E\}.$$

Linear symplectic spaces–2

Definition *A linear subspace E of the symplectic vector space $(V, \tilde{\omega})$ is*
isotropic, if $E \subset E^{\perp_{\tilde{\omega}}}$,
coisotropic, if $E \supset E^{\perp_{\tilde{\omega}}}$,
Lagrangian, if $E = E^{\perp_{\tilde{\omega}}}$,
symplectic, if $E \cap E^{\perp_{\tilde{\omega}}} = \{0\}$.

Example $(\mathbf{R}^{2n}, \omega_0)$ with $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$ is the standard linear symplectic space. $\omega_0(u, v) = v^T J u$ for $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$.
$L_0 = \{0\} \oplus \mathbf{R}^n$ and $L_1 = \mathbf{R}^n \oplus \{0\}$ are two Lagrangian subspaces.
The set of Lagrangian subspaces of $(\mathbf{R}^{2n}, \omega_0)$ is denoted as $\Lambda(n)$.
The symplectic group is defined by

$$Sp(2n) = \{M \in \mathcal{L}(\mathbf{R}^{2n}) |\ M^T J M = J\}.$$

Hamiltonian systems–1

For a smooth function $H(t, x)$ with $(t, x) \in \mathbf{R} \times \mathbf{R}^{2n}$, the nonlinear Hamiltonian system is the following

$$\dot{x}(t) = JH'(t, x(t)).$$

Suppose $x(t)$ is a solution of the above Hamiltonian system, the linearized system at $x(t)$ is the following linear Hamiltonian system

$$\dot{z}(t) = JB(t)z(t), \tag{1}$$

where $B(t) = H''(t, x(t))$.

Hamiltonian systems–2

There is a matrix function $\gamma(t)$ such that any solution $z(t)$ of the system (1) satisfying the initial condition $z(0) = z_0$ is

$$z(t) = \gamma(t)z_0.$$

So the matrix function $\gamma(t)$ should satisfy

$$\dot{\gamma}(t) = JB(t)\gamma(t), \quad \gamma(0) = I_{2n}.$$

Thus we have $\gamma(t)^T J\gamma(t) = J$, *i.e.*, $\gamma(t) \in Sp(2n)$ is a symplectic path starting from the identity.
We denote by

$$\mathcal{P}(2n) = \{\gamma \in C([0,1], Sp(2n)) \mid \gamma(0) = I_{2n}\}$$

the set of all continuous symplectic paths starting from identity.

Hamiltonian systems–3

For a smooth function $H(t,x)$ with $(t,x) \in \mathbf{R} \times \mathbf{R}^{2n}$, and a Lagrangian subspace L, the nonlinear Hamiltonian system with L-boundary value problem is the following problem

$$\begin{cases} \dot{x}(t) = JH'(t,x(t)), \\ x(0) \in L,\ x(1) \in L, \end{cases}$$

Correspondingly we have the following linear Hamiltonian system with the same L-boundary value condition

$$\begin{cases} \dot{z}(t) = JB(t)z(t), \\ z(0) \in L,\ z(1) \in L, \end{cases} \tag{2}$$

There is a method to classify the system (2), which is the Maslov-type index theory with Lagrangian boundary conditions.

Firstly, we consider a special case. Suppose $L = L_0 = \{0\} \oplus \mathbf{R}^n \subset \mathbf{R}^{2n}$ which is a Lagrangian subspace of the linear symplectic space $(\mathbf{R}^{2n}, \omega_0)$. For a symplectic path $\gamma(t)$, we write it in the following form:

$$\gamma(t) = \begin{pmatrix} S(t) & V(t) \\ T(t) & U(t) \end{pmatrix}.$$

where $S(t), T(t), V(t), U(t)$ are $n \times n$ matrices. The n vectors coming from the column of the matrix $\begin{pmatrix} V(t) \\ U(t) \end{pmatrix}$ are linear independent and they span a Lagrangian subspace of $(\mathbf{R}^{2n}, \omega_0)$. Particularly, at $t = 0$, this Lagrangian subspace is $L_0 = \{0\} \oplus \mathbf{R}^n$.

We define the following two subsets of $Sp(2n)$ by

$$Sp(2n)^*_{L_0} = \{M \in Sp(2n) \,|\, \det \mathrm{V_M} \neq 0\},$$

$$Sp(2n)^0_{L_0} = \{M \in Sp(2n) \,|\, \det \mathrm{V_M} = 0\},$$

for $M = \begin{pmatrix} S_M & V_M \\ T_M & U_M \end{pmatrix}$.

$$\mathcal{P}(2n)^*_{L_0} = \{\gamma \in \mathcal{P}(2n) \,|\, \gamma(1) \in Sp(2n)^*_{L_0}\}$$

and

$$\mathcal{P}(2n)^0_{L_0} = \{\gamma \in \mathcal{P}(2n) \,|\, \gamma(1) \in Sp(2n)^0_{L_0}\}.$$

Definition We define the L_0*-nullity of any symplectic path* $\gamma \in \mathcal{P}(2n)$ by

$$\nu_{L_0}(\gamma) \equiv \dim \ker_{L_0}(\gamma(1)) := \dim \ker V(1) = n - \mathrm{rank V}(1) \quad (2.2)$$

with the $n \times n$ matrix function $V(t)$ defined in (2.1).

Example For the linear Hamiltonian system with $B(t) = I_{2n}$:

$$\dot{z}(t) = Jz(t),$$

we have $\gamma(t) = e^{Jt} = \begin{pmatrix} I_n \cos t & -I_n \sin t \\ I_n \sin t & I_n \cos t \end{pmatrix}$. It is an orthogonal symplectic matrix. The $n \times n$ complex matrix from the last n columes is $I_n \cos t + \sqrt{-1} I_n \sin t$ is an unitary matrix with its determinent $e^{n\sqrt{-1}t}$. For the general case, suppose $\gamma \in \mathcal{P}^*_{L_0}(2n)$ with

$$\gamma(t) = \begin{pmatrix} S(t) & V(t) \\ T(t) & U(t) \end{pmatrix}.$$

Then we have

$$\mathcal{Q}(t) = \mathcal{Q}_\gamma(t) = [U(t) - \sqrt{-1}V(t)][U(t) + \sqrt{-1}V(t)]^{-1}$$

is a unitary matrix function. We should modify the end matrices to define the index as an integer.

Maslov-type index theory–4

So We denote by

$$M_+ = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \quad M_- = \begin{pmatrix} 0 & J_n \\ -J_n & 0 \end{pmatrix}, \quad J_n = \mathrm{diag}(-1, 1, \cdots, 1)$$

It is clear that $M_\pm \in Sp(2n)^{\pm}_{L_0} = \{M \in Sp(2n) | \pm \det \mathrm{V_M} > 0\}$, the two path connected components of $Sp(2n)^*_{L_0}$. For a path $\gamma \in \mathcal{P}(2n)^*_{L_0}$, we first adjoin it with a simple symplectic path starting from $J = -M_+$, i.e., we define a symplectic path by

$$\tilde{\gamma}(t) = \begin{cases} I \cos\frac{(1-2t)\pi}{2} + J \sin\frac{(1-2t)\pi}{2}, & t \in [0, 1/2]; \\ \gamma(2t-1), & t \in [1/2, 1] \end{cases}$$

then we choose a symplectic path $\beta(t)$ in $Sp(2n)^*_{L_0}$ starting from $\gamma(1)$ and ending at M_+ or M_- according to $\gamma(1) \in Sp(2n)^+_{L_0}$ or $\gamma(1) \in Sp(2n)^-_{L_0}$, respectively. We now define a joint path by

$$\bar{\gamma}(t) = \beta * \tilde{\gamma} := \begin{cases} \tilde{\gamma}(2t), & t \in [0, 1/2], \\ \beta(2t-1), & t \in [1/2, 1]. \end{cases}$$

Maslov-type index theory–5

As above, we define

$$\bar{\mathcal{Q}}(t) = [\bar{U}(t) - \sqrt{-1}\bar{V}(t)][\bar{U}(t) + \sqrt{-1}\bar{V}(t)]^{-1}. \qquad (2.6)$$

for $\bar{\gamma}(t) = \begin{pmatrix} \bar{S}(t) & \bar{V}(t) \\ \bar{T}(t) & \bar{U}(t) \end{pmatrix}$. We can choose a continuous function $\bar{\Delta} : [0,1] \to \mathbf{R}$ such that

$$\det \bar{\mathcal{Q}}(\mathrm{t}) = \mathrm{e}^{\sqrt{-1}\bar{\Delta}(\mathrm{t})}.$$

Definition For a symplectic path $\gamma \in \mathcal{P}(2n)^*_{L_0}$, we define the *$L_0$-index of* γ by

$$i_{L_0}(\gamma) = \frac{1}{2\pi}(\bar{\Delta}(1) - \bar{\Delta}(0)).$$

Maslov-type index theory–6

Suppose $\lambda_j(t) = e^{\sqrt{-1}\theta_j(t)}$ are the eigenvalues of $\mathcal{Q}_\gamma(t)$ for $j = 1, \cdots, n$.

Proposition *(C. Liu 2007) For $\gamma \in \mathcal{P}(2n)^*_{L_0}$, with the above notations, there holds*

$$i_{L_0}(\gamma) = \sum_{j=1}^{n} \left[\frac{\theta_j(1) - \theta_j(0)}{2\pi} \right].$$

Definition For a symplectic path $\gamma \in \mathcal{P}(2n)^0_{L_0}$, we define the *$L_0$-index of γ* by

$$i_{L_0}(\gamma) = \inf\{ i_{L_0}(\tilde{\gamma}) \,|\, \tilde{\gamma} \in \mathcal{P}^*_{L_0}, \text{and } \tilde{\gamma} \text{ is sufficiently close to } \gamma \}.$$

For any Lagrangian subspace L, we know that the set of Lagrangian subspaces in $(\mathbf{R}^{2n}, \omega_0)$ denoted by $\Lambda(n)$ satisfying $\Lambda(n) = U(n)/O(n)$, this means that for any Lagrangian subspace $L \in \Lambda(n)$, there is an orthogonal symplectic matrix $P = \begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ with $A \pm \sqrt{-1}B \in U(n)$, such that $PL_0 = L$. P is uniquely determined by L up to an orthogonal matrix $C \in O(n)$.

We define the conjugated symplectic path $\gamma_c \in \mathcal{P}$ of γ by $\gamma_c(t) = P^{-1}\gamma(t)P$.

Definition For a symplectic path $\gamma \in \mathcal{P}$, we define the *L-index of γ* by

$$(i_L(\gamma), \nu_L(\gamma)) = (i_{L_0}(\gamma_c), \nu_{L_0}(\gamma_c)).$$

The properties of the L-indices–1

Definition For two symplectic paths $\gamma_0, \gamma_1 \in \mathcal{P}(2n)$, we say that they are L_0-*homotopic* and denoted by $\gamma_0 \sim_{L_0} \gamma_1$, if there is a map $\delta : [0,1] \to \mathcal{P}$ such that $\delta(j) = \gamma_j$ for $j = 0, 1$, and $\nu_{L_0}(\delta(s))$ is constant for $s \in [0,1]$.

Theorem *If $\gamma_0, \gamma_1 \in \mathcal{P}(2n)^*_{L_0}$, then $i_{L_0}(\gamma_0) = i_{L_0}(\gamma_1)$ if and only if $\gamma_0 \sim_{L_0} \gamma_1$.*

Theorem *For any symplectic path $\gamma \in \mathcal{P}(2n)$, by using the notations in above Proposition, there holds*

$$i_{L_0}(\gamma) = \sum_{j=1}^{n} E\left(\frac{\theta_j(1) - \theta_j(0)}{\pi}\right),$$

where $E(a) = \max\{k \in \mathbf{Z} |\ k < a\}$.

The properties of the L-indices–1

Theorem (**Homotopy invariant**) *For two path* $\gamma_j \in \mathcal{P},\ j = 0, 1$, *if* $\gamma_0 \sim_{L_0} \gamma_1$, *there hold*

$$i_{L_0}(\gamma_0) = i_{L_0}(\gamma_1),\ \ \nu_{L_0}(\gamma_0) = \nu_{L_0}(\gamma_1).$$

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix},\ L_0 = L_0' \oplus L_0''.$$

Theorem (**Symplectic addivity**) *For two path* $\gamma_j \in \mathcal{P}(n_j),\ j = 1, 2$, *there holds*

$$i_{L_0}(\gamma_1 \diamond \gamma_2) = i_{L_0'}(\gamma_1) + i_{L_0''}(\gamma_2).$$

L-index for general symplectic paths–1

For a general continuous symplectic path $\rho : [a,b] \to Sp(2n)$ and the Lagrangian subspace L_0.

Definition *We define*

$$\hat{i}_{L_0}(\rho) = i_{L_0}(\gamma_b) - i_{L_0}(\gamma_a), \tag{3}$$

where $\gamma_a \in \mathcal{P}(2n)$ is a symplectic path ended at $\rho(a)$ and $\gamma_b \in \mathcal{P}(2n)$ is the composite of symplectic path γ_a and ρ, i.e., $\gamma_b = \rho * \gamma_a$.
In general, for any Lagrangian subspace L, we define

$$\hat{i}_L(\rho) = i_L(\gamma_b) - i_L(\gamma_a). \tag{4}$$

We remind that for the constant path $\gamma = I$ there holds $i_{L_0}(I) = -n$ and $i_{L_1}(I) = -n$, so $\hat{i}_{L_0}(\gamma) = i_{L_0}(\gamma) + n$ and $\hat{i}_{L_1}(\gamma) = i_{L_1}(\gamma) + n$ for $\gamma \in \mathcal{P}(2n)$.

L-index for general symplectic paths–2

Theorem *The index $\hat{i}_{L_0}$ has the following properties*

1 *(Affine Scale Invariance). For $k > 0$, $l \geq 0$, we have the affine map $\varphi : [a, b] \to [ka + l, kb + l]$ defined by $\varphi(t) = kt + l$. For a given continuous path $\rho : [ka + l, kb + l] \to Sp(2n)$, there holds*

$$\hat{i}_{L_0}(\rho) = \hat{i}_{L_0}(\rho \circ \varphi). \tag{5}$$

2 *(Homotopy Invariance rel. End Points). If $\delta : [0, 1] \times [a, b] \to Sp(2n)$ is a continuous map with $\delta(0, t) = \rho_1(t)$, $\delta(1, t) = \rho_2(t)$, $\delta(s, a) = \rho_1(a) = \rho_2(a)$ and $\delta(s, b) = \rho_1(b) = \rho_2(b)$ for $s \in [0, 1]$, then*

$$\hat{i}_{L_0}(\rho_1) = \hat{i}_{L_0}(\rho_2). \tag{6}$$

Theorem *(Continue Theorem)*

3 *(Path Additivity). If* $a < b < c$, *and* $\rho_{[a,c]} : [a,c] \to Sp(2n)$ *is concatenate path of* $\rho_{[a,b]}$ *and* $\rho_{[b,c]}$, *then there holds*

$$\hat{i}_{L_0}(\rho_{[a,c]}) = \hat{i}_{L_0}(\rho_{[a,b]}) + \hat{i}_{L_0}(\rho_{[b,c]}). \tag{7}$$

4 *(Symplectic Additivity). Let* L_0^k, $\rho_k : [a,b] \to Sp(2n_k)$, $k = 1, 2$, $L_0 = L_0^1 \diamond L_0^2$, $\rho = \rho_1 \diamond \rho_2$. *Then we have*

$$\hat{i}_{L_0}(\rho) = \hat{i}_{L_0^1}(\rho_1) + \hat{i}_{L_0^2}(\rho_2). \tag{8}$$

Here the symplectic direct sum of two Lagrangian subspaces L' *and* L'' *is defined by*

$$L' \diamond L'' = \{(x', x'', y', y'')^T |\ (x', y')^T \in L',\ (x'', y'')^T \in L''\}.$$

Theorem *(Continue Theorem)*

5 *(Symplectic Invarince) Let matrix $P \in Sp(2n)$ be a symplectic matrix. We have*

$$\hat{i}_{PL_0}(P\rho P^{-1}) = \hat{i}_{L_0}(\rho). \tag{9}$$

6 *(Normalization). For $L_0 = \mathbf{R}$ and $\rho : [-\varepsilon, \varepsilon] \to Sp(2)$ with $\varepsilon > 0$ small and $\rho(t) = e^{Jt}$, we have*

$$\begin{array}{l} (i) \;\; \hat{i}_{L_0}(\rho) = 1; \\ (ii) \;\; \hat{i}_{L_0}(\rho_{[-\varepsilon,0]}) = 0; \\ (iii) \; \hat{i}_{L_0}(\rho_{[0,\varepsilon]}) = 1. \end{array} \tag{10}$$

The index for paths of Lagrangian pair–1

We are ready to define an index for a pair of a continuous Lagrangian path $f : [0,1] \to \Lambda(n) \times \Lambda(n)$ with $f(t) = (L_1(t), L_2(t)), 0 \leq t \leq 1$. We know that there are $U_1(t),\ U_2(t) \in \mathrm{Osp}(2n)$ such that $L_j(t) = U_j(t)L_0$, then we have the following definition.

Definition

$$\hat{i}_0(f) = \hat{i}_{L_0}(\gamma_{12}), \tag{11}$$

where $\gamma_{12}(t) = U_1(t)^{-1}U_2(t),\ 0 \leq t \leq 1.$

For a general symplectic space $V = (\mathbb{R}^{2n}, \omega)$ and a path of Lagrangian pair $f : [a,b] \to \mathrm{Lag}(V) \times \mathrm{Lag}(V)$, by choosing a linear symplectic map $T : (\mathbb{R}^{2n}, \omega) \to (\mathbb{R}^{2n}, \omega_0)$, we can define

$$\hat{i}(f) = \hat{i}_0(TfT^{-1}). \tag{12}$$

The index for paths of Lagrangian pair–2

Theorem *The definition is well defined. Furthermore $\hat{i}(f)$ has the following properties*

1. *For $k > 0$, $l \geq 0$, we have the affine map $\varphi : [a, b] \to [ka + l, kb + l]$ defined by $\varphi(t) = kt + l$. For a given continuous path $f : [ka + l, kb + l] \to \mathrm{Lag}(V) \times \mathrm{Lag}(V)$, there holds*

$$\hat{i}(f) = \hat{i}(f \circ \varphi). \tag{13}$$

2. *If $\delta : [0, 1] \times [a, b] \to \mathrm{Lag}(V) \times \mathrm{Lag}(V)$ is a continuous map with $\delta(0, t) = f_1(t)$, $\delta(1, t) = f_2(t)$, $\delta(s, a) = f_1(a) = f_2(a)$ and $\delta(s, b) = f_1(b) = f_2(b)$ for $s \in [0, 1]$, then*

$$\hat{i}(f_1) = \hat{i}(f_2). \tag{14}$$

Theorem *(Continue the Theorem)*

3. *If* $a < b < c$, *and* $f_{[a,c]} : [a, c] \to \mathrm{Lag}(V) \times \mathrm{Lag}(V)$ *is concatenate path of* $f_{[a,b]}$ *and* $f_{[b,c]}$, *then there holds*

$$\hat{i}(f_{[a,c]}) = \hat{i}(f_{[a,b]}) + \hat{i}(f_{[b,c]}). \tag{15}$$

4. *Let* $f_k : [a, b] \to \mathrm{Lag}(V_k) \times \mathrm{Lag}(V_k)$, $k = 1, 2$, $V = V_1 \oplus V_2$ $f = f_1 \oplus f_2$. *Then we have*

$$\hat{i}(f) = \hat{i}(f_1) + \hat{i}(f_2). \tag{16}$$

Theorem *(Continue the Theorem)*

5 *Let* $P(t) \in Sp(2n)$ *be a symplectic path and* $f(t) = (L_1(t), L_2(t)) \in \Lambda(n) \times \Lambda(n)$. *We define* $(P_* f)(t) = (P(t)L_1(t), P(t)L_2(t))$. *Then we have*

$$\hat{i}_0(P_* f) = \hat{i}_0(f). \tag{17}$$

6 *Let* $V_0 = (\mathbf{R}^2, \omega_0)$. *Define* $f : [-\varepsilon, \varepsilon] \to \Lambda(1) \times \Lambda(1)$ *with* $\varepsilon > 0$ *small as a pair Lagrangian path:*

$$f(t) = (\mathbf{R}, \mathbf{R}(\cos t, sint)),\ t \in [-\varepsilon, \varepsilon].$$

Then

$$\begin{array}{ll} (i) & \hat{i}_0(f) = 1, \\ (ii) & \hat{i}_0(f_{[-\varepsilon,0]}) = 0, \\ (iii) & \hat{i}_0(f_{[0,\varepsilon]}) = 1. \end{array} \tag{18}$$

The index for paths of Lagrangian pair–3

Let $V = (\mathbf{R}^{2n}, \omega)$ be a symplectic vector space. The set of its Lagrangian space is $\mathrm{Lag}(V)$. A path of Lagrangian pair f is a continuous map:

$$f : [a, b] \to \mathrm{Lag}(V) \times \mathrm{Lag}(V)$$

for some interval $[a, b]$, $a < b$. So $f(t) = (L_1(t), L_2(t)) \in \mathrm{Lag}(V) \times \mathrm{Lag}(V)$. The set of all paths of Lagrangian pair is denote by $P(V)$.

Theorem *There is a unique function $\mu_V : P(V) \to \mathbf{Z}$ satisfying the above axioms (1)-(6).*

Remark *S. E. Cappell, R. Lee and E. Y. Miller in Comm. Pure Appl. Math. 1994 gave a proof of the above Theorem and constructed four differential index theories satisfying the axioms (1)-(6). So our definition is the fifth defined by differential method.*

The iteration theory–1

Theorem *Suppose $\gamma \in \mathcal{P}_\tau(2n)$, for the iteration symplectic paths γ^k defined in (21)-(22) below, when k is odd, there hold*

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + \sum_{i=1}^{\frac{k-1}{2}} i_{\omega_k^{2i}}(\gamma^2), \tag{19}$$

when k is even, there hold

$$i_{L_0}(\gamma^k) = i_{L_0}(\gamma^1) + i_{\sqrt{-1}}^{L_0}(\gamma^1) + \sum_{i=1}^{\frac{k}{2}-1} i_{\omega_k^{2i}}(\gamma^2), \tag{20}$$

where $\omega_k = e^{\pi\sqrt{-1}/k}$ and $i_\omega(\gamma)$ is the ω index of the symplectic path γ, and the index pair $i_{\sqrt{-1}}^{L_0}(\gamma^1)$ is the L_0-ω index of γ with $\omega = \sqrt{-1}$.

The iteration theory–2

Let $N = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}$, we have the Lagrangian subspace $L_0 = \mathrm{Fixpoint(N)}$. For $\gamma \in \mathcal{P}(2n)$, we define

$$\gamma^{2k-1}(t) = \begin{cases} \gamma(t),\ t \in [0,1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1),\ t \in [1,2], \\ \cdots\cdots \\ N\gamma(2k-2-t)N\gamma(2)^{k-1},\ t \in [2k-3, 2k-2], \\ \gamma(t-2k+2)\gamma(2)^{k-1},\ t \in [2k-2, 2k-1], \end{cases} \tag{21}$$

$$\gamma^{2k}(t) = \begin{cases} \gamma(t),\ t \in [0,1], \\ N\gamma(2-t)\gamma(1)^{-1}N\gamma(1),\ t \in [1,2], \\ \cdots\cdots \\ \gamma(t-2k+2)\gamma(2)^{k-1},\ t \in [2k-2, 2k-1], \\ N\gamma(2k-t)N\gamma(2)^{k},\ t \in [2k-1, 2k]. \end{cases} \tag{22}$$

Applications

For the index theory and its iteration theory, we have the following applications.

1. minimal periodic problem for brake orbits of nonlinear Hamiltonian system.

2. subharmonic solutions of brake orbits of nonlinear Hamiltonian system.

3. Seifert conjecture.

4. Conley conjecture in brake orbits.

Thank you for your attention !

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