Boundedness and quasi-periodic solutions in nonlinear oscillations

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Problem

- J. Littlewood proposed to study the boundedness of solutions for

$$\frac{d^2x}{dt^2} + g(x) = p(t), \tag{1}$$

where $g(x)$ satisfies $xg(x) > 0$, and p is continuous and periodic in t.

- Example:

$$\frac{d^2x}{dt^2} + x = \sin t.$$

- According to the growth of g as $x \to \infty$, the equation (1) is divided into three cases.
- Superlinear: $g(x)/x \to +\infty$.
- Sublinear: $g(x)/x \to 0$.
- Semilinear: $0 < \kappa \le g(x)/x \le K < +\infty$.

Superlinear case

- Moser(1973) proposed to study the same problem for

$$\frac{d^2x}{dt^2} + \alpha x + \beta x^3 = p(t),$$

where α, β are positive constants and p is continuous and periodic in t.

- The firts result obtained by G. Morris(1976), who proved the boundedness of solutions as well as the existence of quasi-periodic solutions for $\alpha = 0, \beta = 2$.
- R. Direckerhoff and E. Zehnder(1987) prove the same result for the following equation

$$\frac{d^2x}{dt^2} + x^{2n+1} + \sum_{j=0}^{2n} p_j(t)x^j = 0,$$

if p_j are smooth.

- M. Levi(1991) studied the following equation

$$\frac{d^2x}{dt^2} + F(t, x) = 0,$$

where F is C^5 in t.

- Later, the smooth assumption on p_j are improved to C^2 by X. Yuan(1998,2017).
- B. Liu (1989, 1992) proved the boundedness of solutions for equation

$$\frac{d^2x}{dt^2} + \alpha(t)x + \beta x^3 = p(t)$$

for continuous α and p.

Sublinear case

- The first result obtained by T. Kupper and J. You(1991), who proved the boundedness of solutions for the equation

$$\frac{d^2x}{dt^2} + |x|^{\alpha-1}x = p(t)$$

where $0 < \alpha < 1$, p is smooth.

- B, Liu(2001) extended this result to general case

$$\frac{d^2x}{dt^2} + g(x) = p(t),$$

where p is also smooth.

- Open question: is there an example for p is only continuous?

Semilinear case

- The boundedness problem is more subtle.
- The first result is due to R. Ortega(1996)
- The Boundedness is obtained by him for the equation

$$\frac{d^2x}{dt^2} + ax^+ - bx^- = 1 + \epsilon p(t) \tag{2}$$

where $a \neq b$ two positive constants, ϵ small parameter, p smooth.

- B. Liu(1998) studies the following equations

$$\frac{d^2x}{dt^2} + n^2x + \phi(x) = p(t), \tag{3}$$

where $\phi(x) \to \pm\infty$ as $x \to \pm\infty$, p smooth.

- There is no small parameter!

- Ortega(1999) also study the above equation (3) under the assumption that ϕ is bounded.

  More precisely, he assume that $\phi(x)$ is piecewise linear function

$$\phi(x)=\begin{cases} -L, & x\le -1,\\ Lx, & -1\le x\le 1,\\ L, & x\ge 1.\end{cases}$$

  He prove that if

$$\left|\frac{1}{2\pi}\int_0^{2\pi} p(t)e^{int}dt\right| < \frac{2L}{\pi}$$

  then all solutions of (3) are bounded.

- B. Liu (1999) drop the Ortega's assumption on ϕ. Under the condition

$$\left|\frac{1}{2\pi}\int_0^{2\pi} p(t)e^{int}dt\right| < \frac{\phi(+\infty)-\phi(-\infty)}{\pi} \tag{4}$$

  we get the same conclusion.
- The inequality (4) is called Lazer-Landesman condition.
- Open Problem: what is happened if

$$\left|\frac{1}{2\pi}\int_0^{2\pi} p(t)e^{int}dt\right| = \frac{\phi(+\infty)-\phi(-\infty)}{\pi}.$$

- There is a result (B. Liu) for

$$\left|\frac{1}{2\pi}\int_0^{2\pi} p(t)e^{int}dt\right| = \frac{\phi(+\infty)-\phi(-\infty)}{\pi} = 0.$$

- B. Liu(2004) also study the same problem when the function ϕ depend on time t.

Isochronous center: sigular case

- We(Capietto and Liu) study the existence of quasi-periodic solutions as well as boundedness of solutions for the equation

$$\frac{d^2x}{dt^2} + V_x(x) = p(t);$$

where p is a π-periodic function and, for $x > -1$

$$V(x) = \frac{1}{2}x_+^2 + \frac{1}{(1 - x_-^2)^\gamma} - 1, \quad \gamma > 0.$$

- isochronous case: B. Liu(2009) considers the boundedness of solutions for isochronous center with singular potential function

$$\frac{d^2x}{dt^2} + V_x(x) + g(x) = p(t).$$

- For example

$$V_x(x) = \frac{x+1}{4} - \frac{1}{4(1+x)^3}.$$

- The lazer-Landesman condition

$$\lim_{\rho\to+\infty}\int_0^{2\pi} g(\rho\left|\sin\frac{t}{2}\right|)\left|\sin\frac{t}{2}\right|dt > \int_0^{2\pi} p(t+\theta)\left|\sin\frac{t}{2}\right|dt$$

  is needed.
- What will be happened if the above inequality is violated?

The tool for the proof: Invariant curves for area-preserving mapping

Consider the mapping

$$\mathcal{M}: \quad \theta_1 = \theta + \alpha(r) + \epsilon f(\theta, r), \quad r_1 = r + \epsilon g(\theta, r),$$

where f, g are smooth functions and $\alpha' \neq 0$.

- Question: Under what conditions on f, g, the mapping $\mathcal{M}$ has invariant curves.
- Example: $g = 1$, there is no invariant curves!
- Condition: $\mathcal{M}$ is area-preserving.
- Moser(1962) proved the existence of invariant curves for f and g smooth.

Theorem

Assume that f and g are bounded in C^p $(p > 3)$. Then there exists $\epsilon_0 > 0$ such that the mapping $\mathcal{M}$ has an invariant curve in the domain $\mathbb{R} \times [a, b]$ if $0 < \epsilon < \epsilon_0$.

- This theorem can be used to prove the boundedness of solutions and the existence of quasi-periodic solutions for superlinear and sublinear cases.
- However, it seems that we cannot use it to study the same problem for semilinear equations.

Ortega(1999) study the following mapping

$$\mathcal{M}_\delta : \begin{cases} \theta_1 = \theta + \beta + \delta l(\theta, r) + \delta f(\theta, r, \delta), \\ r_1 = r + \delta m(\theta, r) + \delta g(\theta, r, \delta), \end{cases} \qquad (\theta, r) \in \mathbb{R} \times [a, b],$$

where the functions l, m, f, g are 2π-periodic in θ, $f(\theta, r, 0) = g(\theta, r, 0) = 0$, β is a constant, $0 < \delta < 1$ is a small parameter.

Theorem

Assume that $\beta/2\pi$ is irrational and $l(\cdot,\cdot), m(\cdot,\cdot), f(\cdot,\cdot,\delta), g(\cdot,\cdot,\delta) \in C^6$ and

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial l}{\partial r}(\theta, r)d\theta \neq 0.$$

Then there exists $\Delta_0 > 0$ such that the mapping $\mathcal{M}_\delta$ has an invariant curve in the domain $\mathbb{R} \times [a, b]$ if $0 < \delta < \Delta_0$.

Theorem

Assume that $\beta = 2n\pi$ *and* $l(\cdot,\cdot), m(\cdot,\cdot), f(\cdot,\cdot,\delta), g(\cdot,\cdot,\delta) \in \mathcal{C}^6$*; furthermore, suppose that*

$$l > 0, \quad \frac{\partial l}{\partial r} > 0,$$

and there is a function $\Phi(\theta, r)$ *such that*

$$\Phi \in C^6, \quad \frac{\partial \Phi}{\partial r} > 0,$$

$$l(\theta, r)\frac{\partial \Phi}{\partial \theta} + m(\theta, r)\frac{\partial \Phi}{\partial r} \equiv 0.$$

Then there exists $\Delta_0 > 0$ *such that the mapping* $\mathcal{M}_\delta$ *has an invariant curve in the domain* $\mathbb{R} \times [a, b]$ *if* $0 < \delta < \Delta_0$.

The above two theorems can be generalized to quasi-periodic mappings.

$$\mathcal{M}_\delta : \begin{cases} \theta_1 = \theta + \beta + \delta l(\theta, r) + \delta f(\theta, r, \delta), \\ r_1 = r + \delta m(\theta, r) + \delta g(\theta, r, \delta), \end{cases} \qquad (\theta, r) \in \mathbb{R} \times [a, b],$$

where the functions l, m, f, g are quasi-periodic in θ with the frequency ω, $f(\theta, r, 0) = g(\theta, r, 0) = 0$, β is a constant, $0 < \delta < 1$ is a small parameter.

Theorem

Assume that $\omega_1, \omega_2, \cdots, \omega_n, 2\pi/\beta$ are rationally independent, $l(\cdot,\cdot), m(\cdot,\cdot), f(\cdot,\cdot,\delta), g(\cdot,\cdot,\delta) \in C^p$ $(p > 2\tau + 1 > 2n + 1)$ and

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \frac{\partial l}{\partial r}(\theta, r) d\theta \neq 0.$$

Then there exists $\Delta_0 > 0$ such that the mapping $\mathcal{M}_\delta$ has an invariant curve in the domain $\mathbb{R} \times [a, b]$ if $0 < \delta < \Delta_0$. The invariant curve is quasi-periodic with the frequency ω.

The functions l and m can be represented in the form

$$\begin{aligned} l(\theta, r) := \tilde{l}(\theta, r) + \bar{l}(\theta, r) &= \sum_{k \in \mathbb{Z}^n \setminus \mathbb{K}} l_k(r) e^{i\langle k, \omega \rangle \theta} + \sum_{k \in \mathbb{K}} l_k(r) e^{i\langle k, \omega \rangle \theta} \\ &= \sum_{k \in \mathbb{Z}^n \setminus \mathbb{K}} l_k(r) e^{i\langle k, \omega \rangle \theta} + \sum_{\ell \in \mathbb{Z}} l_{\ell k^0}(r) e^{(2j_0\pi/\beta) i \ell \theta}, \end{aligned}$$

$$\begin{aligned} m(\theta, r) := \widetilde{m}(\theta, r) + \overline{m}(\theta, r) &= \sum_{k \in \mathbb{Z}^n \setminus \mathbb{K}} m_k(r) e^{i\langle k, \omega \rangle \theta} + \sum_{k \in \mathbb{K}} m_k(r) e^{i\langle k, \omega \rangle \theta} \\ &= \sum_{k \in \mathbb{Z}^n \setminus \mathbb{K}} m_k(r) e^{i\langle k, \omega \rangle \theta} + \sum_{\ell \in \mathbb{Z}} m_{\ell k^0}(r) e^{(2j_0\pi/\beta) i \ell \theta}. \end{aligned}$$

$$\bar{l}(\theta + \beta, r) \equiv \bar{l}(\theta, r), \quad \overline{m}(\theta + \beta, r) \equiv \overline{m}(\theta, r).$$

Theorem

- Suppose that the functions l, m, f, g satisfy

$$l(\cdot,\cdot) \in C^{p+2}\ (p > 2\tau + 1 > 2n + 1), \quad \bar{l}(\theta, r) > 0, \quad \frac{\partial \bar{l}(\theta, r)}{\partial r} > 0,$$

$$m(\cdot,\cdot), \quad f(\cdot,\cdot,\delta), \quad g(\cdot,\cdot,\delta) \in C^{p+1}.$$

- There is a function $L(\theta, r) \equiv L(\theta + \beta, r)$ satisfying

$$L \in C^{p+2}, \qquad \frac{\partial L(\theta, r)}{\partial r} > 0 \tag{5}$$

$$\bar{l}(\theta, r)\frac{\partial L}{\partial \theta}(\theta, r) + \overline{m}(\theta, r)\frac{\partial L}{\partial r}(\theta, r) \equiv 0, \tag{6}$$

and two numbers $\widetilde{a}$ and $\widetilde{b}$ such that

$$a < \widetilde{a} < \widetilde{b} < b$$

and

$$L_{\max}(a) < L_{\min}(\widetilde{a}) \leq L_{\max}(\widetilde{a}) < L_{\min}(\widetilde{b}) \leq L_{\max}(\widetilde{b}) < L_{min}(b), \tag{7}$$

$$L_{\min}(r) := \min_{\theta\in\mathbb{R}} L(\theta, r), \quad L_{\max}(r) := \max_{\theta\in\mathbb{R}} L(\theta, r).$$

Then there exists $\Delta_0 > 0$ such that if $\delta < \Delta_0$, the mapping $\mathcal{M}_\delta$ has an invariant curve which is quasi-periodic with the frequency ω=($\omega_1, \omega_2, \cdots,$ ω_n). The constants Δ_0 depend only on $a,\ b,\ \widetilde{a}, \widetilde{b},\ l(\theta, r), m(\theta, r)$ and $L(\theta, r)$.

Application

Consider the following equation

$$x''+ax^{+}-bx^{-}=f(t), \tag{8}$$

where a, b are two different positive constants, $x^{+} = \max\{x, 0\}$, $x^{-} = \max\{-x, 0\}$, $f(t)$ is smooth quasi-periodic function with the frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$.

Action and angle variables

Introduce a new variable $y = x'$, then (8) is equivalent to the following planar system

$$\begin{cases} x' = y, \\ y' = -ax^{+} + bx^{-} + f(t). \end{cases} \tag{9}$$

Let $C(t)$ be the solution of the initial value problem

$$\begin{cases} x'' + ax^{+} - bx^{-} = 0, \\ x(0) = 1,\ x'(0) = 0. \end{cases}$$

Then it is well known that $C(t) \in \mathcal{C}^2(\mathbb{R})$ which can be given by

$$C(t) = \begin{cases} \cos\sqrt{a}t, & |t| \in [0, \frac{\pi}{2\sqrt{a}}], \\ -\sqrt{\frac{a}{b}}\sin\sqrt{b}\left(|t| - \frac{\pi}{2\sqrt{a}}\right), & |t| \in [\frac{\pi}{2\sqrt{a}}, \frac{\pi}{2\sqrt{a}} + \frac{\pi}{2\sqrt{b}}]. \end{cases}$$

Define $S(t)$ be the derivative of $C(t)$, then $S(t) \in \mathcal{C}^1(\mathbb{R})$ and
(i) $C(-t) = C(t), S(-t) = -S(t)$.
(ii) $C(t)$ and $S(t)$ are $2\omega_0\pi$-periodic functions, $\omega_0 = \frac{1}{2}(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}})$.

(iii) $S^2(t) + a(C^+(t))^2 + b(C^-(t))^2 \equiv a$.
For $r > 0$, $\theta \pmod{2\pi}$, we define the following generalized polar coordinates $T : (r, \theta) \to (x, y)$ as

$$\begin{cases} x = \varrho r^{\frac{1}{2}} C(\omega_0 \theta), \\ y = \varrho r^{\frac{1}{2}} S(\omega_0 \theta), \end{cases}$$

where $\varrho = \sqrt{\frac{2}{a\omega_0}}$. It is easy to check that T is a symplectic transformation.

system (9) is changed into the following generalized polar coordinate system

$$\begin{cases} \theta' = \omega_0^{-1} - \frac{1}{2}\varrho C(\omega_0\theta)f(t)r^{-\frac{1}{2}}, \\ r' = \omega_0\varrho S(\omega_0\theta)f(t)r^{\frac{1}{2}}. \end{cases} \tag{10}$$

Now we change the role of the variable t and θ, and yields that

$$\begin{cases} \frac{dt}{d\theta} = \left[\omega_0^{-1} - \frac{1}{2}\varrho C(\omega_0\theta) f(t) r^{-\frac{1}{2}}\right]^{-1}, \\ \frac{dr}{d\theta} = \omega_0 \varrho S(\omega_0\theta) f(t) r^{\frac{1}{2}} \left[\omega_0^{-1} - \frac{1}{2}\varrho C(\omega_0\theta) f(t) r^{-\frac{1}{2}}\right]^{-1}, \end{cases} \tag{11}$$

which is 2π-periodic in the new time variable θ. Let r_* be a positive number such that

$$\omega_0^{-1} - \frac{1}{2}\varrho r_*^{-\frac{1}{2}} |C||f| > 0.$$

System (11) is well defined for $r \geq r_*$. Let $\big(t(\theta), r(\theta)\big)$ be a solution of (11) defined in a certain interval $I = [\theta_0, \theta_1]$ and such that $r(\theta) > r_*$ for all θ in I. The derivative $\frac{dt}{d\theta}$ is positive and the function t is a diffeomorphism from I onto $J = [t_0, t_1]$, where $t(\theta_0) = t_0$ and $t(\theta_1) = t_1$. The inverse function will be denoted by $\theta = \theta(t)$. It maps J onto I.

The expression of the Poincaré map of (11)

The Poincaré map P of (11) has the expansion

$$P : \begin{cases} t_1 = t_0 + 2\omega_0\pi + \frac{1}{2}\omega_0^2\varrho\, r_0^{-\frac{1}{2}} \int_0^{2\pi} C(\omega_0\theta) f(t_0 + \omega_0\theta) d\theta + O(r_0^{-1}), \\ r_1^{\frac{1}{2}} = r_0^{\frac{1}{2}} + \frac{1}{2}\omega_0^2\varrho \int_0^{2\pi} f(t_0 + \omega_0\theta) S(\omega_0\theta) d\theta + O(r_0^{-\frac{1}{2}}). \end{cases}$$

- Assume that f has the following Fourier series

$$f(t) = \sum_k f_k e^{i\langle k,\omega\rangle t}.$$

- $$L(t_0) = \int_0^{2\pi} f(t_0 + \omega_0\theta)C(\omega_0\theta)d\theta \neq 0, \quad \text{for all} \quad t_0 \in \mathbb{R}, \tag{12}$$

Theorem

If $f(t) \in C^p$ $(p > 2n+1)$ satisfies (12), $\langle k, \omega\omega_0\rangle \notin \mathbb{Z}$ for any $k \in \mathbb{Z}^n \backslash \{0\}$. Then system (9) has infinitely many quasi-periodic solutions and all solution are bounded.

Theorem

If $f(t) \in C^p$ $(p > 2n+3)$ satisfies (12), $\langle k, \omega\omega_0\rangle \in \mathbb{Z}$ for some $k \in \mathbb{Z}^n$. Denote by $\mathbb{K}$ the lattice of $\mathbb{Z}^n$ such that $\langle k, \omega\omega_0\rangle \in \mathbb{Z}$ for $k \in \mathbb{K}$ and $\langle k, \omega\omega_0\rangle \notin \mathbb{Z}$ for $k \notin \mathbb{K}$, by $f_{\mathbb{K}}(t)$ the function

$$f_{\mathbb{K}}(t) = \sum_{k\in\mathbb{K}} f_k e^{i\langle k,\omega\rangle t}.$$

Moreover, if

$$\Phi_{\mathbb{K}}(t_0) = \frac{\omega_0^2\varrho}{2}\int_0^{2\pi} f_{\mathbb{K}}(t_0+\omega_0\theta)C(\omega_0\theta)d\theta \neq 0, \quad \text{for all} \quad t_0 \in \mathbb{R}.$$

Then system (9) has many quasi-periodic solutions and all solutions are bounded.

Thank you!